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ADAPTIVE FINITE ELEMENT TECHNOLOGY IN INTEGRATED DESIGN AND ANALYSIS

B. A. Szabo, P. K. Basu, D. A. Dunavant and D. Vasilopoulos

January 1981

Prepared for:
Aeroelastic Optimization Office
NASA Langley Research Center
Hampton, VA 23665

CENTER FOR COMPUTATIONAL MECHANICS

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1. EXECUTIVE SUMMARY

The report concludes that it is possible to realize very substantial savings in the analysis part of the computational process in integrated design and analysis of aircraft structures through the use of adaptive finite element technology based on the p-version of the finite element method.

1.1 Background:

Adaptivity is a procedure for efficient reduction of error on the basis of data already computed. The procedure is comprised of two parts: determination of the sources of error in the computed data and estimation of their magnitude, and improvement of the accuracy of approximation by changing the number and/or distribution of the degrees of freedom.

Currently there are only two adaptive finite element computer programs in existence: FEARS, developed at the University of Maryland and COMET, developed at Washington University. The two programs are based on completely different approaches: whereas FEARS improves solution accuracy by selective mesh refinement, COMET upgrades the displacement approximation over finite elements through the addition of progressively higher order hierarchic shape functions.*

This project involved some of the principal developers of both COMET and FEARS: Professor Ivo Babuska of the University of Maryland served as consultant to the project group at Washington University.

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*The term "p-version" is defined in Section 3 (p. 9).

1.2 Main Results

Adaptivity, based on the p-version of the finite element method will impact on integrated design and analysis in two ways: (a) substantial reduction in the cost of analysis and (b) increased reliability of the computed data. These points are discussed in the following:

1.2.1 Cost Reduction

The most important opportunity for cost reduction is offered by the fact that the p-version exhibits faster rate of convergence than the conventional h-version based on uniform or quasiuniform meshes. Typically, for comparable accuracy (say one to five percent error in strain energy), the number of degrees of freedom required in the p-version is only one-fifth to one-tenth the number required in the h-version. This is especially important in design optimization where a large number of analyses must be performed. The required computer time (cost) is roughly proportional to the square of the number of degrees of freedom. Thus $\frac{1}{5}$ to $\frac{1}{10}$ reduction in the number of degrees of freedom results in $\frac{1}{25}$ to $\frac{1}{100}$ reduction in the required computer time in each analysis cycle. It is noted that when the geometry is complicated and a large number of elements is required, the projected savings are similar due to the fact that the accuracy of analysis is most strongly influenced by the number and type of geometric constraint conditions (singularities) which tend to increase with the geometrical complexity of the problem.

Additional savings can be realized from optimizing the distribution of the degrees of freedom. This requires that the relative contribution of each element to the total error of analysis be known. The degrees of freedom are optimally distributed when the contribution of each finite

element to the total error of approximation is the same. (This is the basis for adaptive mesh refinement in FEARS, however the same indicators cannot be used in the p-version). Indicators were developed for the p-version for problems in two-dimensional elasticity under the present project.

The savings that can be realized from optimal distribution of the degrees of freedom are highly problem dependent: in those cases where the number of singularities (reentrant corners, stiffener connections, changes in support conditions) is large in relation to the number of finite elements, the savings will be small. In those cases where the number of singularities is small in relation to the number of elements, the savings can be very substantial.

Computational experiments were conducted with the objective to determine whether still further cost reductions can be realized through the use of the adaptive process in the following way: The number of degrees of freedom (hence the accuracy of approximation) was increased gradually as the extremum was approached by the optimizer. The results of these experiments indicate that the savings are only marginal. The reason is that in statically indeterminate structures the local optima may shift considerably as the accuracy of analysis is increased. Thus the optimizer may not be approaching the correct extremum value initially, which negates the advantage gained from operating with fewer degrees of freedom at the beginning.

A quantitative statement of error tolerance requirements generated in each optimization cycle by the optimizer could, conceivably, result in more substantial savings from this approach.

1.2.2 Reliability

For obvious practical reasons, the finite element model should represent the prototype reliably. In other words, the accuracy of finite element analysis should not be sensitive to input data, such as material properties, type of boundary conditions and loading. In order to verify that a finite element solution is in fact correct, the user must ascertain that the solution is in the asymptotic range either with respect to $h_{max} \to 0$ or $p_{min} \to \infty$. (h_{max} is the maximum size of finite elements, p_{min} is the lowest polynomial order). Because it can be extremely costly to verify that a given finite element solution is indeed in the asymptotic range with respect to $h_{max} \to 0$, such verification is almost invariably omitted in practical analyses based on the h-version of the finite element method. In a number of cases, however, the accuracy of approximation shows extreme sensitivity to material properties and the kind of plate or shell theories used in the h-version, but not so in the p-version. Thus the p-version improves reliability in two ways: by making it simple to check whether a solution is in the asymptotic range with respect to $p_{min} \to \infty$ and by showing almost no sensitivity in the accuracy of analysis to variations in input data.

1.3 Recommendations

In order to minimize computational costs associated with integrated design and analysis processes, the following are recommended.

(i) The analyzer should have p-version capabilities. It is now a well established fact that the p-version of the finite element is substantially more efficient than the conventional (h-version) in the three most costly areas of the analysis process: input data preparation and modification, computer time, and verification of computed results.

(ii) Because the cost of analysis very strongly depends on error tolerance requirements, the maximum acceptable relative error should be determined by the optimizer for all constrained variables and communicated to the analyzer at each computational cycle.

(iii) Development of direct error estimation techniques for all constrained variables should be vigorously pursued. This is a research problem rooted mostly in applied mathematics. The groundbreaking work has been completed, the existence of certain kinds of estimators has been proven. The objective of future research should be to extend the work for all constrained variables (deflections, moments, vibration frequencies) and problem types (stiffened plates and shells) encountered in structural synthesis.

(iv) Until direct error estimation techniques are available, indirect methods should be used. This involves the establishment of benchmark problems, such as those presented in this report, and development of correlations between the type of problem and the smoothness of approximating function on one hand, and relative error in the quantities of interest on the other.

(v) For structural synthesis involving plates and shells finite element models based on the Reissner-Mindlin theory are recommended. It has been demonstrated that the Reissner-Mindlin theory is capable of approximating the corresponding elasticity solutions to within one and three percent relative error in energy for a wide range of thicknesses.

2. INTRODUCTION

There has been very substantial progress in the development of structural optimization methods in recent years. Unfortunately, their acceptance in design practice has been rather slow because efficient integration of general purpose structural analysis systems with general purpose optimizers has not yet been achieved. An approach, which promises to remove this obstacle, was proposed in reference [1]. In this approach, a programming system consisting of a general purpose analysis program, a general purpose optimization program and two problem-dependent interface programs is constructed. A specific implementation, called PROgramming System for Structural Synthesis (PROSSS) has been documented in [2].

In structural synthesis the structure is modified and analyzed repeatedly until the optimality criteria are satisfied. The main cost source in the process is the computer time required for structural analysis. Thus, to improve the efficiency of structural synthesis, it is necessary to reduce the cost of analysis.

Important developments have occurred in this area. In particular, the p-version of the finite element method has been developed and convergence rate theorems have been established for it. These theorems indicate that in most problems of practical interest the number of degrees of freedom required for a given level of precision is substantially smaller in the p-version than in the conventional h-version which is based on uniform or quasi-uniform mesh refinement. Also, it has been demonstrated in the course of the present project that the accuracy of analysis in the p-version is not sensitive to input data such as material parameters and the types of plate and shell theories used to construct

the elemental stiffness matrices and load vectors. Techniques, such as reduced integration, which introduce uncertainties concerning the quality of approximation, are not needed in conjunction with the p-version even if Poisson's ratio is as large as 0.4999 (near incompressibility) or when shear deformation is accounted for in the analysis of thin plates.

The convergence rate theorems are a priori estimates of error i.e. the estimators do not require that the computations be actually performed. These theorems are useful for making general comparisons between alternative modeling strategies. It is desirable to obtain more accurate error estimates however, applicable to specific approximations. Error estimators based on computed results are called a posteriori estimators. It is reasonably certain that reliable a posteriori error estimators can be developed for the p-version of the finite element method, but some theoretical problems remain. The importance of such estimators cannot be overstated. They are essential for attaining the goal of computations which is either to achieve some desired level of precision at minimum overall cost or to obtain the best possible approximation for a fixed cost with a reliable estimate of the error of approximation. The cost cannot be minimized if, due to lack of a reliable error estimator, the analyst is forced to employ more degrees of freedom than necessary in order to be "on the safe side". It is self evident that the quality of an approximation cannot be accurately assessed if means for error estimation are not available. For these reasons, a great deal of attention was given to a posteriori error estimation in the present project. Significant progress has been made in establishing that reliable local a posteriori error indicators exist in the p-version at least for one and two-dimensional elliptic boundary value problems.

In this report we first summarize the main mathematical theorems which establish the p-version of the finite element method. The theorems give a priori error estimates. We then discuss practical considerations resulting from the theorems with respect to computational efficiency. The discussion is supported by examples which are suitable for benchmark comparisons with other computer codes. We also present examples of structural optimization. Finally, the development of a posteriori error indicators is discussed.

3. DEFINITIONS AND PRELIMINARIES

There are two basic convergence processes in finite element analysis. The conventional finite element convergence process in which the number of interpolating (shape) functions are fixed for each element and the mesh is refined in such a way that the maximum diameter of elements, h, approaches zero is called h-convergence. In the other basic convergence process, called p-convergence, the number and distribution of finite elements is fixed and the interpolating functions, which are complete polynomials of order p, optionally supplemented with other types of interpolating functions, is progressively increased.

There are very substantial differences between the two convergence processes in numerical performance and computer implementation. For this reason we refer to the conventional computer implementation of the finite element method in which improvement of accuracy is achieved by mesh refinement as the h-version of the finite element method. Computer implementations in which improvement of accuracy is achieved by increasing the polynomial order over a fixed mesh are referred to as the p-version of the finite element method.

In the interest of programming and computational economy, the p-version utilizes exactly and minimally conforming hierarchic finite elements. Hierarchic finite elements have the property that the set of basis functions of an element of order p is a subset of the basis functions of all higher order elements of the same kind. Consequently, the stiffness matrix of an element is embedded in the stiffness matrices of all higher order elements of the same kind.

Exact and minimal conformity is important for the following reasons: Finite element approximation is based on minimizing the total potential

energy $\pi(\underline{u})$ which, in the absence of body forces, initial stresses and strains that would not affect the following discussion, can be written as

$$\pi(\underline{u}) = \frac{1}{2} \int_{vol} ([D]\underline{u})^T [E]([D]\underline{u}) dV - \int_{S_T} \underline{u}^T \underline{T} \, ds \qquad (3.1)$$

in which [D] is the differential operator matrix that defines the strain-displacement (or equivalent) relations, $\underline{u}$ is the displacement vector field, [E] is the matrix that defines the stress-strain relations (Hooke's law or equivalent), $\underline{T}$ is the surface traction vector. The first integration is over the entire volume of the structure, the second is over that part of the surface for which surface tractions are specified.

The theorem of minimum potential energy is usually stated in the following way: "Of all displacements satisfying the given boundary conditions, those which satisfy the equilibrium equations make the potential energy an absolute minimum". (See, for example, reference 3).

In finite element analysis we are interested in the converse of this theorem, namely, we wish to minimize the potential energy functional $\pi(\underline{u})$ over some set of <u>admissible</u> trial functions $\underline{u}_{FE}$ which will then approximately satisfy the equilibrium equations. It is well known that the admissible functions must satisfy the principal boundary conditions [3]. It is almost self evident that the set of admissible functions should be as large as possible, in order to permit us to make the error $||\underline{u}-\underline{u}_{FE}||$ (measured in some suitable norm, to be discussed later) as small as possible. To explain: if we arbitrarily restricted the class of admissible trial functions, we certainly would not have

improved our chances for minimizing the error in the most efficient way possible. For this reason it is desirable that the interelement continuity requirements be minimally satisfied: Any continuity requirement over the minimum imposes an unnecessary restriction on the class of admissible trial functions.

Let us now examine the consequences of not satisfying the minimal continuity requirements exactly. The continuity requirements arise from the criterion that admissible trial functions, in our case $\underline{u}_{FE}$, must result in finite strain energy. Thus, any trial function for which the first integral in eq. (3.1), called the strain energy $U(\underline{u}_{FE})$, is infinitely large, is not admissible. Therefore $\underline{u}_{FE}$ is admissible if it satisfies the principal boundary conditions and its first partial derivatives are square integrable on the solution domain. In mathematical notation, $\underline{u}_{FE}$ is admissible if $\underline{u}_{FE} \in H^1_\Gamma(\Omega)$. The notation reminds one of the essential characteristics of $\underline{u}_{FE}$: Each component of vector $\underline{u}_{FE}$ belongs to the space of functions H, which are defined on the solution domain Ω, satisfy the principal boundary conditions on the boundaries of Ω, denoted by Γ, and have square integrable first derivatives.

It is not difficult to show that if the trial function $\underline{u}_{FE}$ is not continuous from one finite element to the next then $U(\underline{u}_{FE})$ is unbounded. For example, let us consider the interelement boundary between element A and element B in Fig. 3.1.

For the sake of simplicity let us assume that $u_B(o,y) - u_A(o,y) > 0$ in the range $y_1 \leq y \leq y_2$. Let $M_A(\varepsilon)$ represent the least upper bound (supremum) of u_A in the region $-\varepsilon/2 \leq x \leq 0$, $y_1 \leq y \leq y_2$ and let $M_B(\varepsilon)$

represent the greatest lower bound (infimum) of u_B in the region $0 \leq x \leq \varepsilon/2$, $y_1 \leq y \leq y_2$, $\varepsilon > 0$ arbitrarily. Then:

$$\lim_{\varepsilon \to 0} \int_{y_1}^{y_2} \int_{-\varepsilon/2}^{\varepsilon/2} \left(\frac{\partial u_{FE}}{\partial x}\right)^2 dxdy \geq \lim_{\varepsilon \to 0} \int_{y_1}^{y_2} \int_{-\varepsilon/2}^{\varepsilon/2} \left(\frac{M_B - M_A}{\varepsilon}\right)^2 dxdy = \infty \tag{3.2}$$

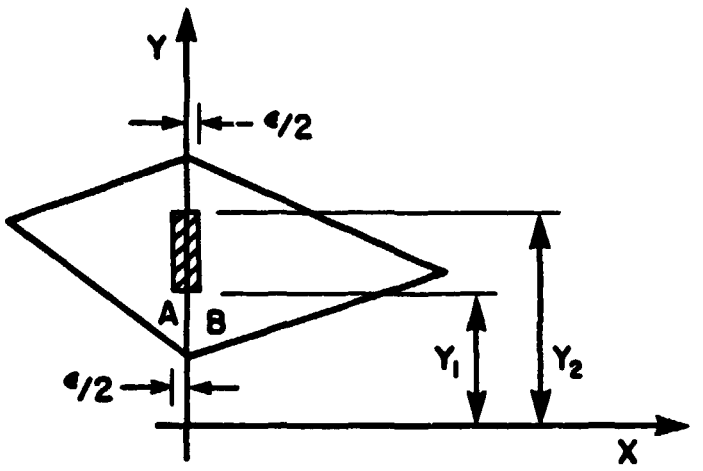


Fig. 3.1

Notation for equation 3.2

Because $(\frac{\partial u_{FE}}{\partial x})^2$ occurs in the strain energy expression, the strain energy is not defined for discontinuous u_{FE}. Nonconforming elements are not admissible. This does not mean that nonconforming elements are not capable of yielding "good results". It does mean that the numerical performance of nonconforming elements is problem-dependent: It is possible to load nonconforming elements in such a way that they deform without absorbing strain energy. Consequently the question of whether the results obtained by means of nonconforming elements are reliable in any given situation must be addressed. We avoid this problem by employing exactly and minimally conforming finite elements and the displacement formulation.

References 4 to 8 present detailed descriptions of exactly and minimally conforming hierarchic finite elements.

3.1 Rates of convergence: uniform or quasi-uniform mesh refinement and p-distribution

We shall now summarize the main results concerning the rates of convergence of the finite element method. For further details, references 9, 10, 11 should be consulted.

The relationship between error and the number of degrees of freedom in finite element analysis is governed by a property of the approximated function, called "smoothness". With some restrictions to be noted later, it is generally true that the smoother the approximated function, the fewer the number of variables needed to achieve a given level of precision. Angular corners at external boundaries tend to reduce smoothness. In two-dimensional elasticity, for example, the approximated displacement vector function is of the form:

$$\underline{u} = r^{\alpha}\, \underline{g}(\theta) + \underline{G}(r,\theta) \qquad (3.1.1)$$

in which r and θ are polar coordinates with the origin at the corner; $\alpha > 1/4$ is a smoothness parameter. (The smoothness of $\underline{u}$ increases with α). Its value depends on the corner angle and the boundary conditions imposed at the corner; $\underline{g}$ and $\underline{G}$ are smoother functions than $\underline{u}$ in the neighborhood of $r = 0$. The values of α for plane elastic and plate problems were given by Williams [12].

When the number of degrees of freedom, N, is increased through uniform or quasi-uniform mesh refinement (h-version), the strain energy of the error is bounded by:

$$U(\underline{u}-\underline{u}_{FE}) \leq \frac{c}{N^{\min(p,\alpha)}} \qquad (3.1.2)$$

in which U is the strain energy; c is a constant which depends on the order p of the polynomial approximation, the value of elastic constants and the geometry of the mesh.

A sequence of mesh refinements is <u>quasi-uniform</u> if in the refinement process the ratio of largest element diameter to the smallest tends to a finite number.

When the number of degrees of freedom is increased through increasing the order of polynomial approximation (p-version), the strain energy of the error is bounded by:

$$U(\underline{u}-\underline{u}_{FE}) \leq \frac{k}{N^{2\alpha}} \qquad (3.1.3)$$

in which k is a constant which depends on the elastic constants, the geometry of the mesh and the distribution of polynomial orders over the mesh. This relationship was established only very recently [10].

The exponents of N in eqs. 3.1.2 and 3.1.3 are called the asymptotic rates of convergence. When log $U(\underline{u}-\underline{u}_{FE})$ is plotted against log N, then, according to eqs. 3.1.2 and 3.1.3, for sufficiently large N, a straight line is obtained whose slope is the asymptotic rate of convergence. When the energy of the error of a given finite element solution is plotted against N on a log-log scale, and it falls on the straight line portion, then the finite element solution is in the asymptotic range.

Naturally, we want our finite element solution to be reliable. This means that the error should not depend significantly on input parameters such as the choice of mesh, polynomial order or material properties but rather it should depend only on the function being approximated. For this reason a finite element solution should be in the asymptotic range and when the p-version or the h-version with uniform or quasi-uniform mesh refinement is used, it should be governed by the parameter α.

The foregoing summary touches on some of the most important characteristics of the finite element method. In order to underline and illustrate the main points, a simple example is presented.

3.2 Example: Parabolically Loaded Square Panel

The parabolically loaded square panel is illustrated in Fig. 3.2.1. The approximated displacement vector field $\underline{u}$ is exceptionally smooth in this case. Specifically, the value of α (defined in eq. 3.1.1) is 2.542, whereas in practial problems α is usually between 0.5 and 0.75.

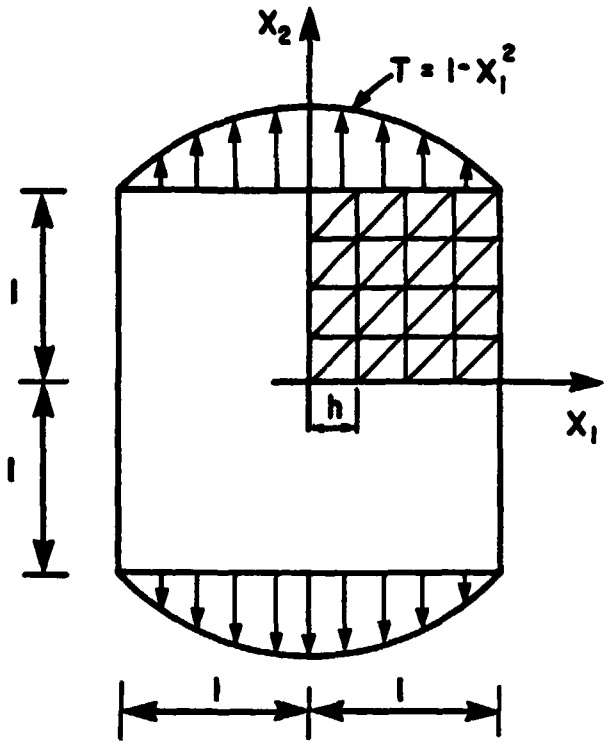


Fig. 3.2.1

Parabolically loaded square panel
Typical finite element mesh

The problem was chosen to demonstrate that in smooth problems the polynomial order controls the rate of h-convergence until $p \geq \alpha$.

The results for Poisson's ratio of 0.3 are shown in Fig. 3.2.2. For $p = 1$ and $p = 2$ the asymptotic range is entered at about 50 degrees of freedom in the h-version. The slopes for $N > 50$ (i.e. the rates of convergence) are governed by p, as predicted by eq. (3.1.2). For $p = 3$ the rate of convergence is governed by α, again as predicted by eq. (3.1.2). Because of the exceptional smoothness of the approximated function, the relative errors are exceptionally small. The faster rates of p-convergence for the different meshes are shown by the dashed lines.

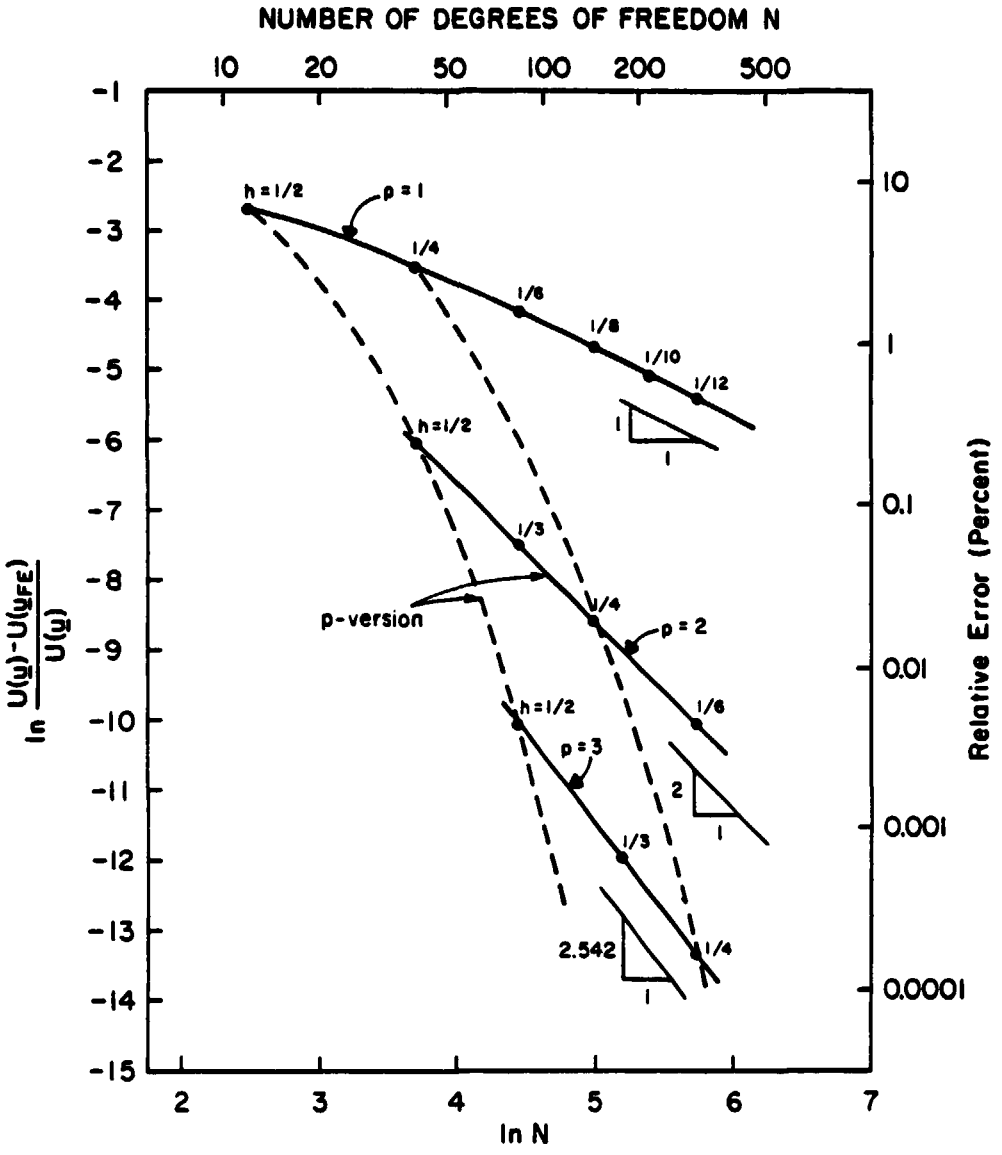


Fig. 3.2.2

Parabolically loaded square panel
Relative error in strain energy vs. N. Poisson's ratio: 0.3

The results for Poisson's ratio of 0.4999 (near incompressibility) are shown in Fig. 3.2.3. For p = 1 we observe that the pre-asymptotic rate of convergence is extremely slow. In fact the asymptotic range is not entered until the roundoff limitations of most digital computers are reached. This example shows that when the solution is substantially

unchanged when the number of degrees of freedom is increased, does not necessarily mean that the solution "has converged". In this case it has not even entered the asymptotic range.

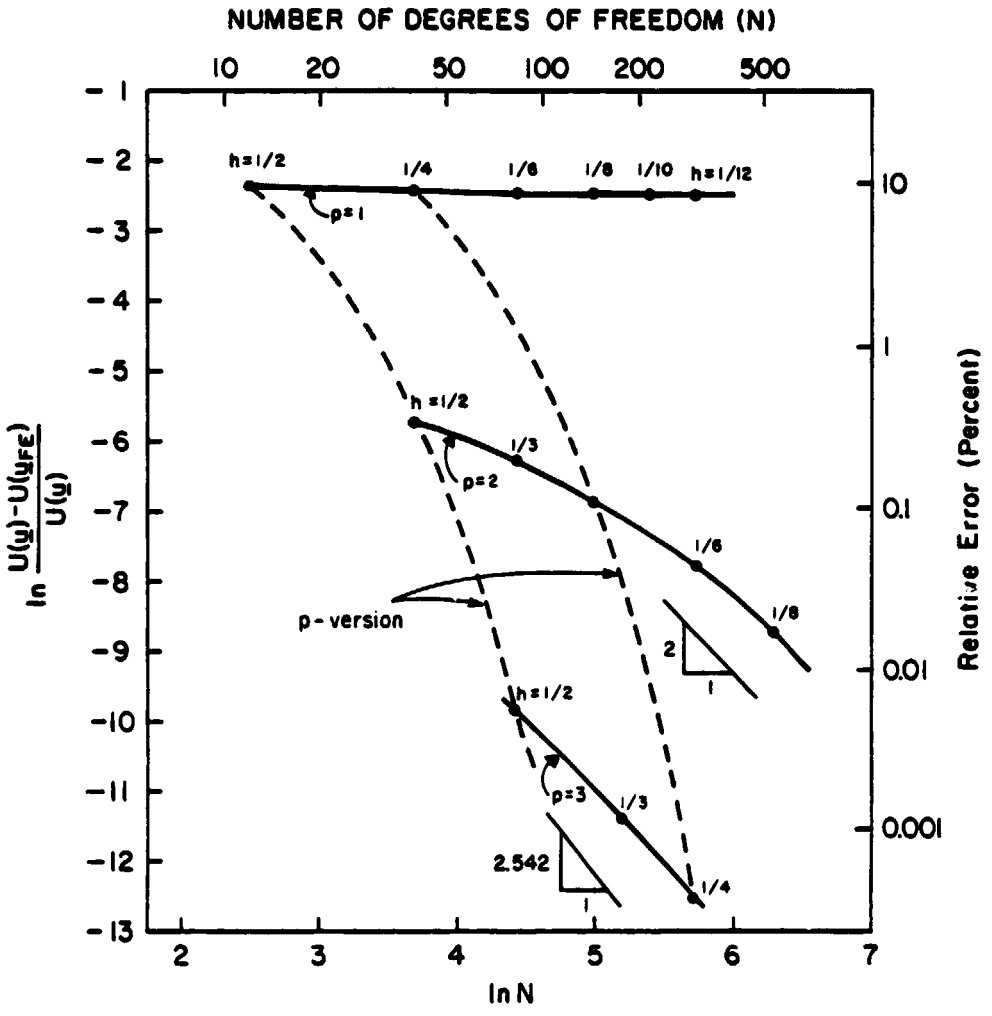


Fig. 3.2.3

Parabolically loaded square panel
Relative error in strain energy vs. N. Poisson's ratio: 0.4999.

For p = 2 the pre-asymptotic rate of convergence is slower than the asymptotic rate. For p = 3 the pre-asymptotic and asymptotic rates are essentially the same.

For the p-version the much faster rate of convergence for the different meshes is indicated by the dashed lines.

The poor convergence characteristics of the h-version in the case of nearly incompressible solids are well known (see, for example [13]). The reduced integration technique, which raises new questions relating to quality control, has been proposed. The p-version eliminates the need for reduced integration.

The behavior of h-convergent approximations is similar in the case of thin plates and shells based on Mindlin's theory: When shear is included in the strain energy expression, its significance in relation to the flexural strain energy progressively diminishes as the thickness is reduced. Mathematically, a penalty multiplier is applied to the shear strain energy which increases, relative to the flexural strain energy, in proportion to t^{-2}, t being the plate thickness. This is analogous to the case of nearly incompressible solids, in which the penalty multiplier is a function of Poisson's ratio and is applied to that part of the strain energy which is associated with volumetric strain.

3.3 Rates of Convergence: Non-quasiuniform Mesh

Substantial research effort has been devoted to finding ways to increase the rate of convergence of the h-version of the finite element method. The most important result of this effort is that it has been proven in [14] that there exist sequences of non-quasiuniform meshes such that the rate of convergence in the h-version is governed by the polynomial order p, not by the corner singularities [i.e. parameter α in Eq. (3.1.1)]. Thus with proper mesh refinement, the rate of convergence is:

$$U(\underline{u}-\underline{u}_{FE}) \leq \frac{k}{N^p} \tag{3.3.1}$$

rather than the rate given by eq. (3.1.2) which is valid for uniform and quasiuniform mesh refinements. The proper mesh refinement is problem dependent, however. This means that the meshes must be determined adaptively. The adaptive mesh refinement procedure is based on error estimators. The estimators measure the error contribution of each element to the total error of approximation. The mesh is then refined so as to make the error contribution from each element as nearly uniform as possible. The procedure has been implemented in a computer code called FEARS (Finite Element Adaptive Research Solver) at the University of Maryland [15].

The finding that the asymptotic rate of h-convergence can be made independent from the corner singularities and in fact arbitrarily large, since the polynomial order p can be chosen arbitrarily, is certainly one of the most important theoretical results concerning the convergence characteristics of the finite element method. It is doubtful, however, that this result can be exploited for purposes of quality control in general purpose finite element computer programs. The algorithmic structure of adaptive finite element codes based on the h-version is very complicated due to the fact that the mesh changes at each adaptive step. The cost of data management operations can substantially exceed the cost of solution of the linear systems of equations. Furthermore, adaptivity in itself does not alleviate the sensitivity of the h-version to input parameters such as Poisson's ratio and plate thickness.

In the p-version the asymptotic rate of convergence cannot be made independent from the corner singularities by adaptively selected p-distributions but it is possible to reduce the number of degrees of freedom without reducing the accuracy by optimizing the p-distribution. The basis for obtaining optimal p-distribution is similar to that used in the h-version: Error indicators are computed which measure the relative contribution of each element to the total error of approximation. The p-distribution is optimal when the error contribution of each element is the same. At present the relative error contributions can be estimated for plane elastic problems from the residuals obtained when the finite element solution is substituted into the Navier equations. This development will be discussed in detail in section 6. Error estimators, similar to those in [14] are not yet available for the p-version.

4. THE COST OF ACCURACY

The goal of computations in engineering practice in general and structural synthesis in particular is either to achieve some desired level of precision at minimum overall cost or to obtain the best possible approximation for a fixed cost with a reliable estimate of the error of approximation.

The desired level of precision is typically around 1 percent in strain energy and 1 to 3 percent in stress resultants (moments and shear forces); stress intensity factors and vibration frequencies in linear analyses. It is not meaningful to measure the error in terms of stress maxima because due to the almost invariably present corner singularities the maximum stress computed on the basis linear elasticity is infinitely large.

The overall cost of computations is comprised of three major parts: Data preparation, computer time and verification of results. Because data preparation usually involves large amounts of human effort, it is generally regarded as the largest of the three cost items. The cost of computer time is diminishing in relation to the others, nevertheless in certain types of analyses, such as optimization and nonlinear analyses, computer time remains the controlling factor. State of the art (h-version) computer codes do not provide methods for verification of computed data which are accepted in general on the basis of the analysts' professional judgement developed through experimentation with benchmark problems. Such indirect methods of verification are not always reliable in the h-version, however. For example, we have seen in the case of nearly incompressible solids that no change in the strain energy with respect

to increasing the degrees of freedom by mesh refinement did not guarantee the solution error to be small. The reason for this is that the pre-asymptotic rate of convergence of the h-version is sensitive to input parameters, in this case to Poisson's ratio.

We shall now consider how the costs of structural analyses may be reduced and the reliability increased by means of existing or proven technology.

4.1 Comparison of the h and p versions in the asymptotic range

Let us first assume that finite element solutions were obtained by both the h- and p-versions for a given problem and the approximations are in the asymptotic range. Let us compare, under these conditions, the computer costs of increasing accuracy in the computed strain energy by one significant digit, using uniform mesh refinement and p-distribution, and the costs involved in structural synthesis.

We note that the asymptotic convergence rate for the relative error in strain energy, e, defined as:

$$e = \frac{U(\underline{u}-\underline{u}_{FE})}{U(\underline{u}_{FE})} \tag{4.1.1}$$

is the same as asymptotic rates given by eqs. 3.1.2 and 3.1.3. Denoting the relative error corresponding to d significant digits of precision by e_d, we have, by definition:

$$\frac{e_{d+1}}{e_d} = 10^{-1} \tag{4.1.2}$$

The major variable cost in the solution of finite element problems is associated with solving the corresponding system of simultaneous equations. In general:

$$C \sim N^{\beta} \tag{4.1.3}$$

in which C is the cost of computer time and β is a number between 1 and 3. β depends on the bandedness or the front width of the linear system of equations and consequently on whether the h- or p-version is used. Dependence on the mode of convergence is quite weak, however, and can be ignored for the purposes of the present discussion. In two-dimensional applications the usual value of β is close to 2. For example, the half band width B of the structure stiffness matrix corresponding to a uniform mesh on a square domain is proportional to $N^{1/2}$. The number of operations required for solving the linear system of equations is proportional to NB^2, hence $\beta = 2$. In computational experiments involving p-convergent approximations β was found to be close to 2 also when Irons' frontal solution technique was utilized. Thus, although the stiffness matrix tends to be more fully populated in the p-version than in the h-version, sparse matrix solution techniques provide substantial reduction in the number of operations as compared with solvers that do not account for sparsity ($\beta = 3$). We remark that only the number of internal modes increases in proportion with p^2. The internal modes are eliminated at the element level. The number of external modes, which represent connectivity among finite elements and must therefore be eliminated at the global level, increases only in proportion to p. On the other hand, both the

external and internal modes increase in proportion to h^{-2}. This tends to compensate for the more heavily populated stiffness matrices in the p-version.

From the rates of convergence and cost formulas (eqs. 3.1.2, 3.1.3 and 4.1.3) we find that the cost is related to the error as:

$$C \sim e^{-\beta/\min(p,\alpha)} \quad \text{in the h-version} \tag{4.1.4}$$

and

$$C \sim e^{-\beta/2\alpha} \quad \text{in the p-version} \tag{4.1.5}$$

Denoting the cost corresponding to d significant digits of precision by C_d, we obtain

$$\frac{C_{d+1}}{C_d} = 10^{\beta/\min(p,\alpha)} \quad \text{in the h-version} \tag{4.1.6}$$

and

$$\frac{C_{d+1}}{C_d} = 10^{\beta/2\alpha} \quad \text{in the p-version} \tag{4.1.7}$$

In linear elastic fracture mechanics, for example, $\alpha = 1/2$. Assuming that $\beta = 2$, we find that <u>the cost increases by the factor of 10,000 for each additional significant digit in the h-version and by a factor of 100 in the p-version</u>.

For structural synthesis, the difference in computer costs between the h- and p-versions based on uniform mesh refinement and p-distribution, can be estimated as follows: Let us consider a typical error versus degrees of freedom diagram, as shown in Fig. 4.1.1. The diagram indicates that two extensions of an initial solution (representing the minimum number of finite elements needed to define the geometry and the lowest polynomial order) were obtained, one by the h-version, the other by the p-version, until the relatve error e was reduced to some

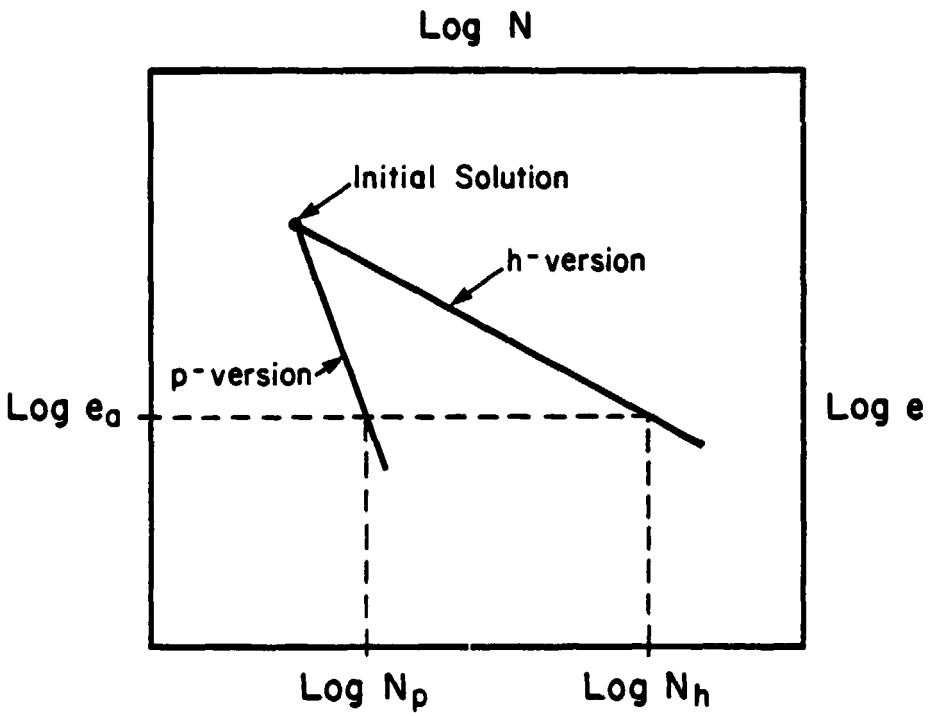


Fig. 4.1.1

Typical relative error vs. number of degrees of freedom diagram representing refinement of an initial solution by the h-version (assuming uniform mesh refinement) and the p-version (assuming uniform p-distribution) in the asymptotic range.

acceptable level e_a. The corresponding degrees of freedom are denoted, respectively, by N_h and N_p. As long as the mesh refinement is uniform or quasiuniform and singularities occur only at the vertices of finite elements, $N_h < N_p$. Since the cost is approximately proportional to the square of the number of degrees of freedom, the cost differential for large N, denoted by ΔC, can be estimated as:

$$\Delta C \sim n(N_h^2 - N_p^2) \tag{4.1.8}$$

in which n is the number of times the analyses is called in the synthesis process. We give specific examples of cost savings in structural synthesis in this report. We emphasize that the estimate represented by (4.1.8) is valid for large N values only.

4.2 Practical Considerations

The foregoing analysis was based on the assumption that both the h- and p-versions are in the asymptotic range. It has been observed by several investigators, however, that in practical analyses the asymptotic range is very seldom entered in the h-version. This means that the solution error is governed by input parameters rather than the true solution of the problem. The pre-asymptotic rate of convergence may be faster or slower than the asymptotic rate. In the p-version on the other hand the asymptotic range is entered at low values of p and, very importantly, the point of entry is substantially independent from the input parameters.

In view of these facts, let us now ask the following question: Given that an analysis has been performed with a minimum number of

finite elements and low polynomial order, and it is desired to improve the accuracy of the solution; which is the better strategy to follow: mesh refinement or increasing the polynomial order? The answer must be based on two considerations: cost and reliability. Both considerations suggest that increasing the polynomial order is the better strategy. The costs of data preparation are certainly less in the case of the p-version since mesh refinement is not involved. The cost of computer time is also less because fewer degrees of freedom are needed to achieve the same level of accuracy in the p-version than in the h-version. And, finally, the p-version is more reliable than the h-version because the entry point into the asymptotic range is not significantly affected by the input parameters.

Compared on the basis of computer costs, the p-version has a very strong advantage over the h-version within the range of accuracies required in engineering analyses. The writers' experience suggests that even when very severe corner singularities are present, such as in linear elastic fracture mechanics, the relative error in strain energy can be reduced to under 1 percent by using coarse meshes and polynomial orders ranging from 6 to 8. For less severe corner singularities the polynomial orders are typically in the range of 3 to 5 for 1 percent error. If, for any reason, the accuracy of analysis must be increased such that the relative error is substantially below 1 percent, the best strategy is to combine the h- and p-versions by using strongly graded fixed meshes at corner singularities and increasing p until the desired level of precision is reached. An example is presented in [9]. Theoretically it is possible to increase both the accuracy and the rate of convergence beyond any limit by

optimal mesh grading coupled with optimal p-distributions. The need for such techniques in practical applications is not foreseen, however.

The cost of verification of computed data in the p-version by means of repeated analyses is further reduced by the hierarchic property of the shape functions: Suppose that an analysis is to be repeated using selectively or uniformly increased polynomial orders. Because the already triangulated stiffness matrix is embedded in the new stiffness matrix, the elimination process must be continued only for the new rows and columns to complete the new analysis. This feature minimizes the marginal cost of verification by repeated analyses.

4.3 Example: Double-edge cracked square panel

We shall demonstrate on the basis of the double edge cracked square panel shown in Fig. 4.3.1 that engineering accuracy can be achieved with the p-version even when severe singularities are present. As in the case of the parabolically loaded square panel, the problem has been analyzed for two different Poisson's ratios ($\nu = 0.3$ and $\nu = 0.4999$). The problem was also analyzed by the h-version. The results for $\nu = 0.3$ are shown in Fig. 4.3.2. It is seen that a coarse mesh, consisting of only 8 elements, and p ranging from 3 to 6 is adequate for obtaining an approximation for which the relative error in energy is between 1 and 5 percent. We note that the error in strain energy is the logical measure for error because the stress intensity factor is related to the strain energy release rate. In fact the rate of convergence of the strain energy release rate has been demonstrated to be the same as the rate of convergence of the strain energy [16].

The h-version, based uniform mesh refinement, requires a very large number of elements to reduce the relative error to 5 percent. When adaptive (non-quasiuniform) mesh refinement is used (Fig. 4.3.3), the rate of convergence of the h-version is governed by eq. 3.3.1. In this case p=1 and 2α=1 hence the same convergence rate is predicted as for the p-version by eq. 3.1.3.

When Poisson's ratio approaches 0.5, the point of entry into the asymptotic range shifts toward very high N values for the h-version but not for the p-version. This is shown in Fig. 4.3.4. The results are similar to those obtained for the parabolically loaded square panel (Fig. 3.2.3). This degradation occurs in connection with the h-version even when adaptive mesh refinement is used.

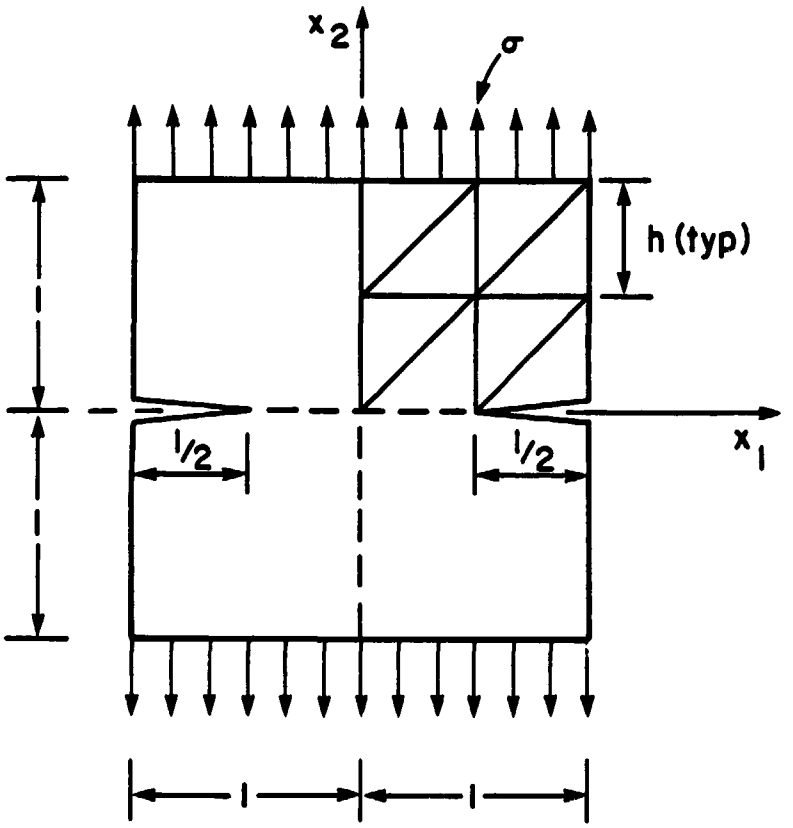


Fig. 4.3.1

Double edge cracked square panel
Initial finite element mesh

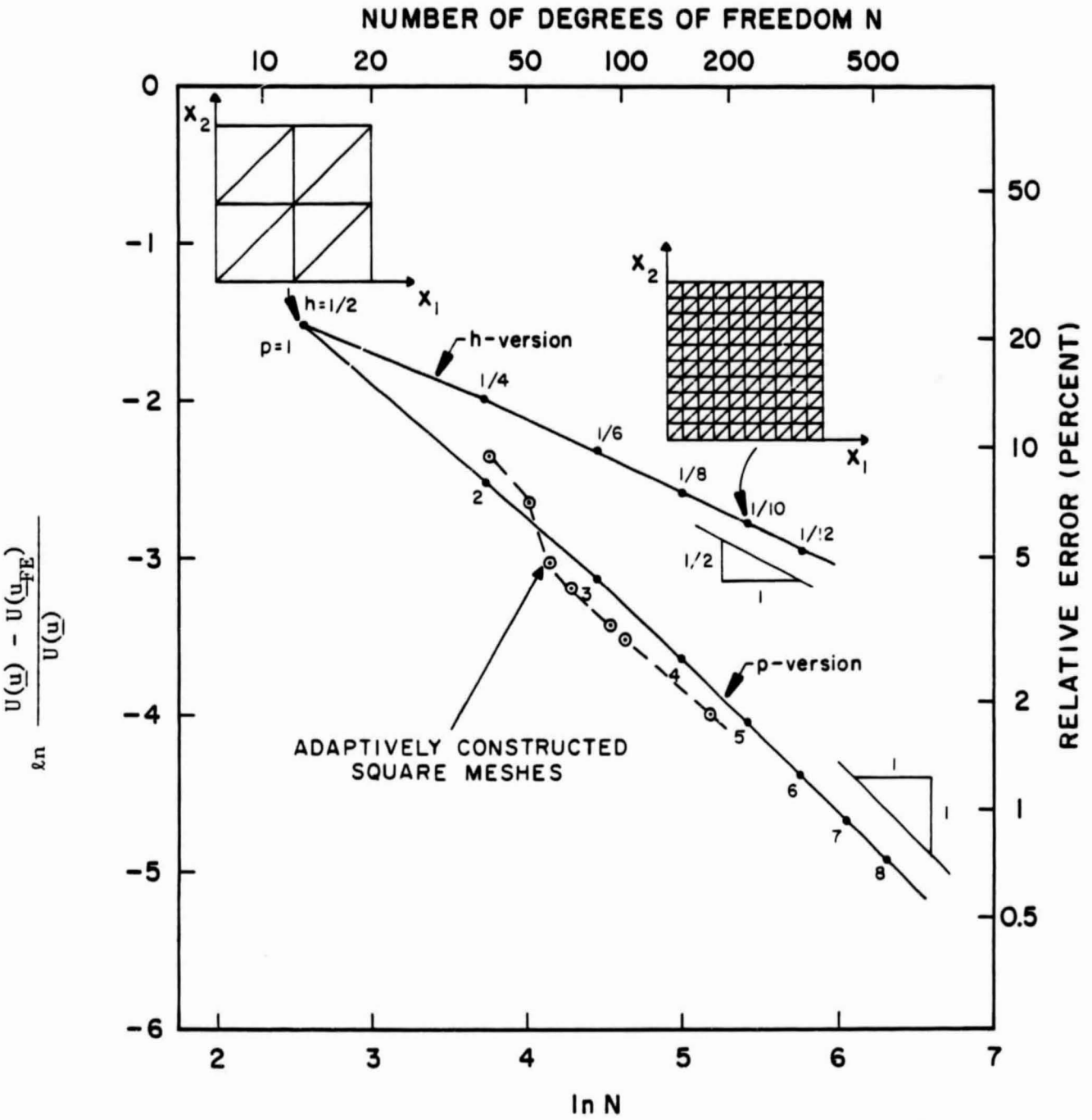
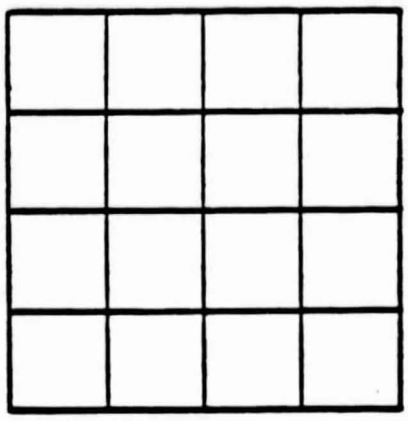
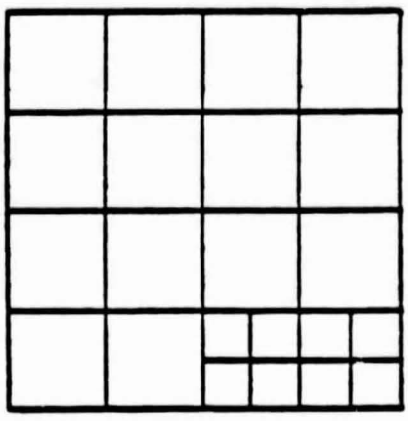


Fig. 4.3.2

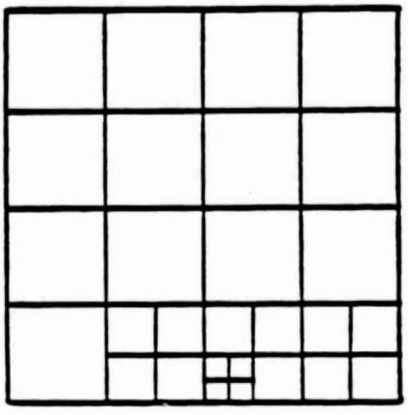
Double edge cracked square panel
Relative error in strain energy vs. N.
Poisson's ratio : 0.3.



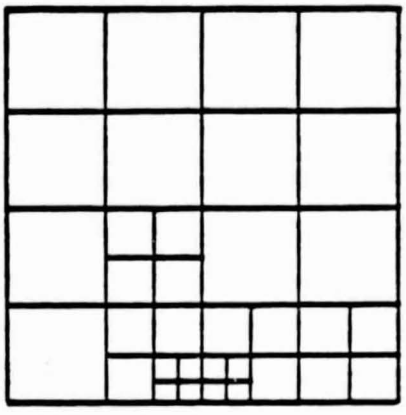
N = 42



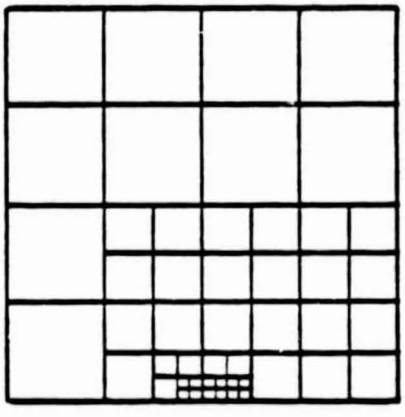
N = 54



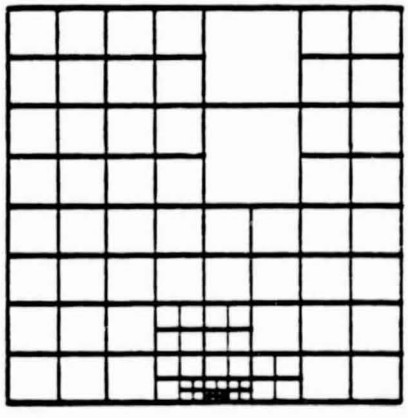
N = 63



N = 72



N = 101



N = 180

Fig. 4.3.3

Double edge cracked square panel.
Sequence of adaptive (non-quasiuniform) meshes
generated by FEARS.

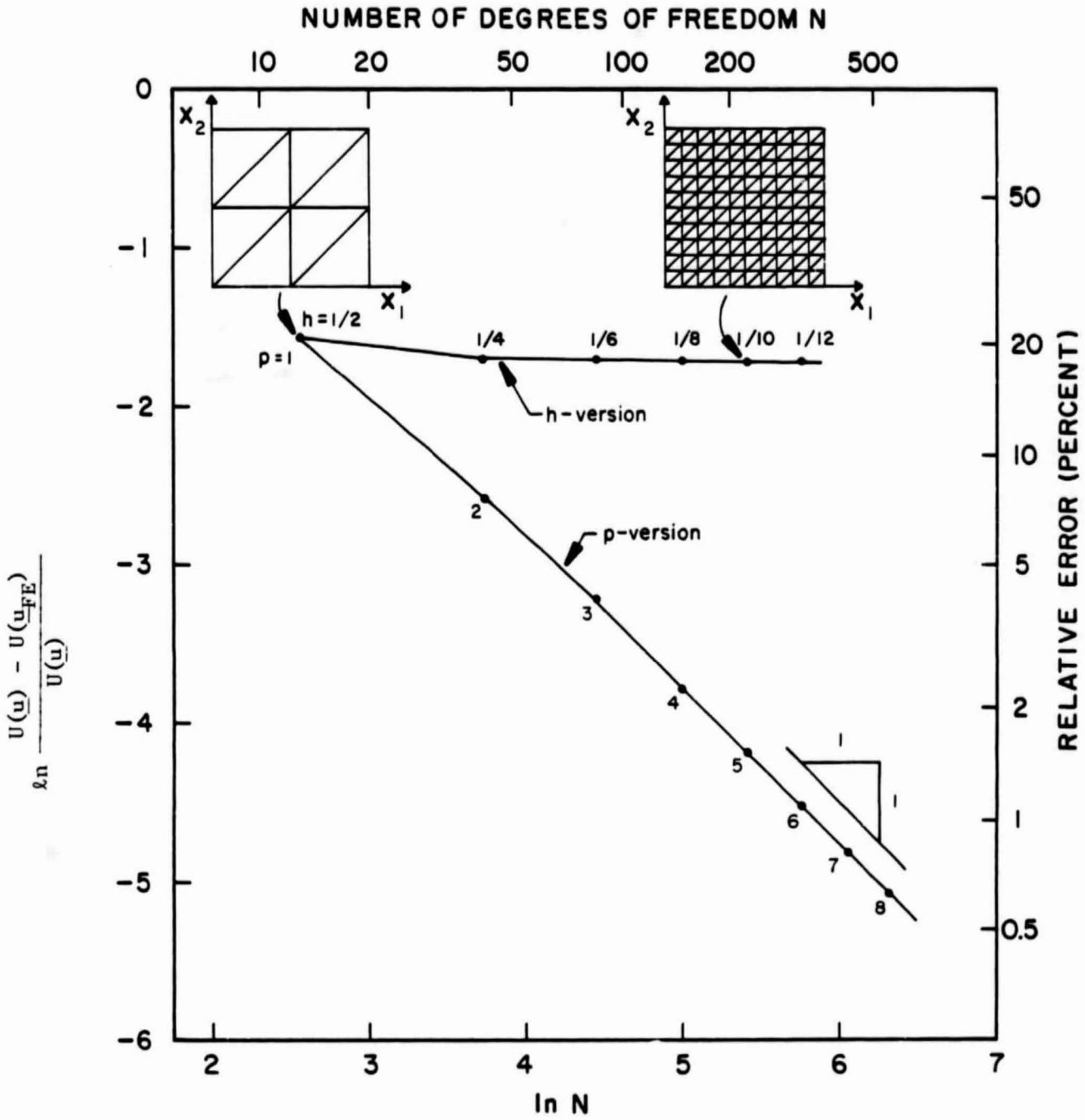


Fig. 4.3.4

Double edge cracked square panel
Relative error in strain energy vs. N.
Poisson's ratio : 0.4999.

5. DIMENSIONAL REDUCTION: VALIDITY OF PLATE BENDING MODELS

A great deal of work has been devoted in the past twenty years to the development of plate bending finite elements. The papers on this subject are too numerous to mention but the problem itself and possible solutions of the problem are adequately described in references 17, 18, 19. Most of the work in this area has been concerned with the enforcement of C^1 continuity which is required for modeling plate behavior on the basis of Kirchhoff's theory.

We approach the problem from a different point of view: Emphasizing that the goal of computations is to approximate the solution of the equations of three dimensional elasticity over the plate domain, the thickness of which is usually much smaller than its other dimensions, we demonstrate that this goal is best achieved through a formulation which is based on the Reissner-Mindlin plate theory [20,21] and the p-version of the finite element method. This formulation has very important advantages:

(i) It is unnecessary to distinguish among "thin", "moderately thick" and "thick" plates because the numerical performance of the formulation is insensitive to plate thickness. (Property of robustness).

(ii) The treatment of curved boundaries does not pose unusual problems because only C^0 continuity is required. C^0 continuity is preserved in transformations to curvilinear coordinates (such as in isoparametric transformation).

(iii) The rate of p-convergence is very rapid. The number of degrees of freedom required for achieving engineering accuracy

(1 to 3 percent error in energy) is generally small even when severe corner singularities are present.

We begin our analysis with a review of the Reissner-Mindlin plate theory and its finite element implementation.

5.1 The Reissner-Mindlin Plate Bending Element:

For the sake of simplicity, let us consider an infinitely long plate strip of width L and thickness t loaded and supported in such a way that the problem can be viewed as a plane strain as well as a plate bending problem (fig 5.1.1). Let us compare, under these conditions, the formulations of the Reissner-Mindlin plate theory and the three-dimensional theory of elasticity.

The assumed displacement functions in the Reissner-Mindlin theory are:

$$u_1 = -x_3\phi(x_1)$$

$$u_2 = 0 \qquad (5.1.1)$$

$$u_3 = \psi(x_1)$$

The assumed displacement functions in the three-dimensional theory of elasticity are:

$$u_1 = \sum_{i=1}^{\infty} -x_3^{2i-1}\,\phi_{2i-1}(x_1)$$

$$u_2 = 0 \qquad (5.1.2)$$

$$u_3 = \sum_{i=0}^{\infty} x_3^{2i}\,\psi_{2i}(x_1)$$

in which the functions $u_1(x_1,x_2)$, $u_3(x_1,x_2)$ have been written as sums of products of powers of the transverse variable x_3 and the field functions $\phi_{(2i-1)}(x_1)$ and $\psi_{(2i)}(x_1)$. We note that the Reissner-Mindlin theory retains only the leading terms of the series which spans the solution space, i.e. is capable of approximating the elasticity solution with arbitrary precision.

The strain-displacement relations are the same in both cases:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5.1.3}$$

The stress-strain relations differ significantly, however. In the case of elasticity:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\,\varepsilon_{ij} \tag{5.1.4}$$

in which λ and G are the Lame parameters.

In the Reissner-Mindlin plate theory the additional assumption is introduced that $\sigma_3 = 0$. We note that this assumption contradicts the assumption implied in eg. (5.5.1), namely that the transverse strain ε_{33} is zero. Consequently, the stress-strain relationship is:

$$\sigma_{11} = \frac{E}{1-\nu^2}\,\varepsilon_{11} \tag{5.1.5}$$

$$\sigma_{22} = \frac{\nu E}{1-\nu^2}\,\varepsilon_{22}$$

$$\sigma_{33} = 0$$

$$\sigma_{13} = \frac{KE}{2(1+\nu)} \varepsilon_{13}$$

In which K is the shear factor, to be discussed in Section 5.2.

In both cases the strain energy expression is based on the fundamental expression:

$$U = \frac{1}{2} \int_{vol} \varepsilon_{ij}\sigma_{ij} dV \qquad (5.1.6)$$

However, because of the differences in the stress-strain relationship, the strain energy expression of the Reissner-Mindlin plate theory will not be the same as the strain energy expression of the theory of elasticity truncated so that only the first beams are retained for u_1 and u_3.

The question arises whether it is reasonable to expect solutions of plate problems obtained by means of the Reissner-Mindlin theory to remain close to corresponding solutions of the three dimensional theory of elasticity when the plate thickness is large with respect to the other dimensions of the plate; with respect to the minimal wave length of oscillatory loading or, more generally, with respect to the minimum wave length of the approximated function. This question has been investigated on the basis of examples presented in sections 5.3, 5.4 and 5.5. The conclusion is that within the range of engineering accuracies (one to three percent error in energy) the Reissner-Mindlin theory can replace the equations of the 3-dimensional theory of elasticity at least for homogeneous isotropic plates. The case of anisotropic plates

had not been investigated. The studies have shown that solutions obtained from the Reissner-Mindlin theory are not sensitive to the shear factor.

5.2 The shear factor

The most complete engineering analysis of the shear factor to date has been presented by Cowper [22]. The shear factors proposed by various authors for rectangular sections range between 0.667 to 0.870. The most commonly used value is 0.833 (5/6). The differences arise from various corrections for the obviously crude approximation of the shear distribution inherent in the Reissner-Mindlin theory. In his analysis Cowper integrated the equations of the three-dimensional theory of elasticity to obtain a formula for the shear factor which accounts for Poisson's ratio:

$$K_c = \frac{10(1+\nu)}{12+11\nu} \tag{5.2.1}$$

A recent mathematical analysis of the shear factor problem was obtained by Vogelius [23]. This analysis is based on asymptotic expansion (with respect to plate thickness) of the equations of elasticity and accounts for transverse load variations with wave length of the order of the plate thickness. Neglecting the effect of rapid transverse load variations, Vogelius' formula is:

$$K_v = \frac{20(1+\nu)}{24+15\nu} \tag{5.2.2}$$

We note that for $\nu=0$ both formulas yield the same shear factor. For $\nu = 0.3$ $K_c = 0.850$ whereas $K_v = 0.912$.

5.3 Example: Infinite plate strip, imposed shear displacement

Let us consider an infinitely long plate strip of width L and thickness d subjected to imposed shear displacement, as shown in Fig. 5.3.1. This problem was chosen for the purpose of studying the effect of shear factor on the error of the plate bending solution in relation to the plane strain solution.

The problem can be viewed as a plane strain problem and also as a plate bending problem. The finite element meshes and principal boundary conditions are shown in Fig. 5.3.1. The plane strain solution was obtained with 6 finite elements, quadratically graded toward the support, and $p = 8$. Advantage was taken of the two axes of antisymmetry: only one quarter of the domain was modelled. The number of degrees of freedom for the quarter domain was 399.

The strain energy of the elasticity solution was estimated by extrapolation on the basis of eq. 3.1.3. The results for various L/d ratios are shown in Table 5.3.1.

Table 5.3.1

Strain energy per unit length of the plane strain solution (U_E).

6 elements, $p = 8$, $E = 1.0$, $\nu = 0.3$, $L = 2$. (One half of the plate only).

L/d	U_E
2	0.7648×10^{-3}
5	0.7846×10^{-4}
10	1.0702×10^{-5}
20	1.3668×10^{-6}
40	1.7163×10^{-7}
200	1.3744×10^{-9}

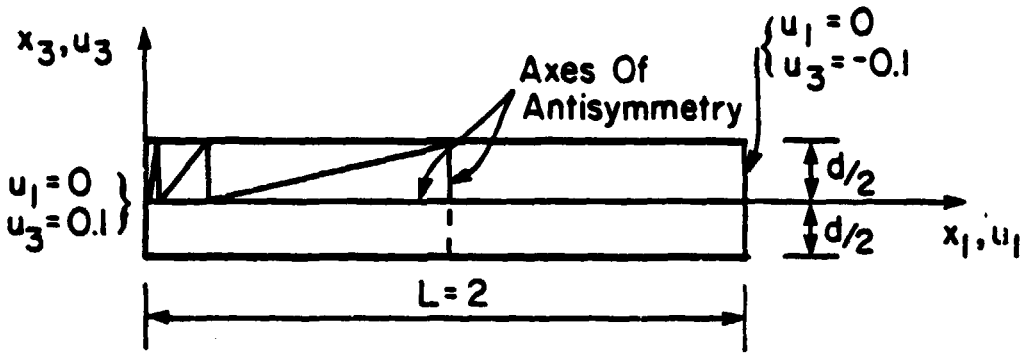


Fig. 5.3.1a

Infinite plate strip. Plane strain model.

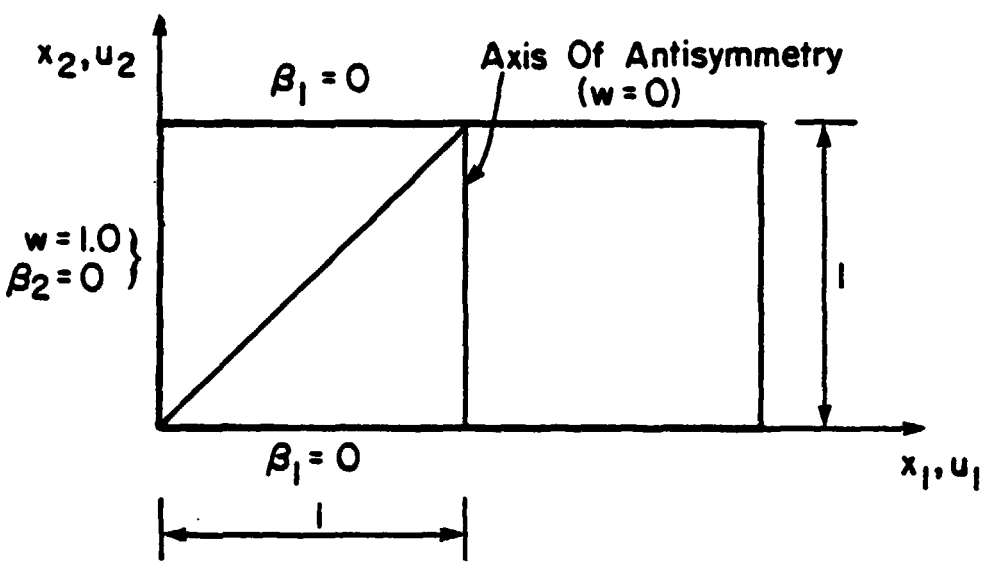


Fig. 5.3.1b

Infinite plate strip. Plate bending model
β_i represents rotation about axis x_i.

The plate bending model consisted of two finite elements covering half of the plate domain, as shown in Fig. 5.3.1.b. The polynomial order was 3, the number of degrees of freedom was 26 for half of the plate domain. (This compared with 399 degrees of freedom used for the quarter domain in the plane strain solution). The plate bending solution did not change as p was increased to 4, 5 etc.

The ratios of strain energies computed from the plate bending model based on the Reissner-Mindlin theory (U_B) and from the plane strain model (U_E) are plotted against the length to thickness ratios for three different shear factors in Fig. 5.3.2. It is seen that for large L/d ratios U_B/U_E is very close to one (the relative error is nearly zero) for all shear factors. For small L/d ratios the shear factor proposed by Vogelius [23] gives the best results. Even when L/d is only 2, the relative error corresponding to Vogelius' shear factor is less than one percent. A large number of studies were performed on the effectiveness of the shear factor in more complicated loading situations as well; for example: oscillatory loading with the wave length of oscillations being equal to the plate thickness. In all cases the results were similar to the results presented here.

The relative errors in maximum deflection, moment and shear force were found to be similar in magnitude to the relative error in strain energy.

This example shows that solutions obtained via the Reissner-Mindlin plate theory and the p-version of the finite element method are efficient and sufficiently accurate for engineering purposes for a very wide range of length to thickness ratios. Vogelius' shear factor is preferable when the length to thickness ratio is less than 20.

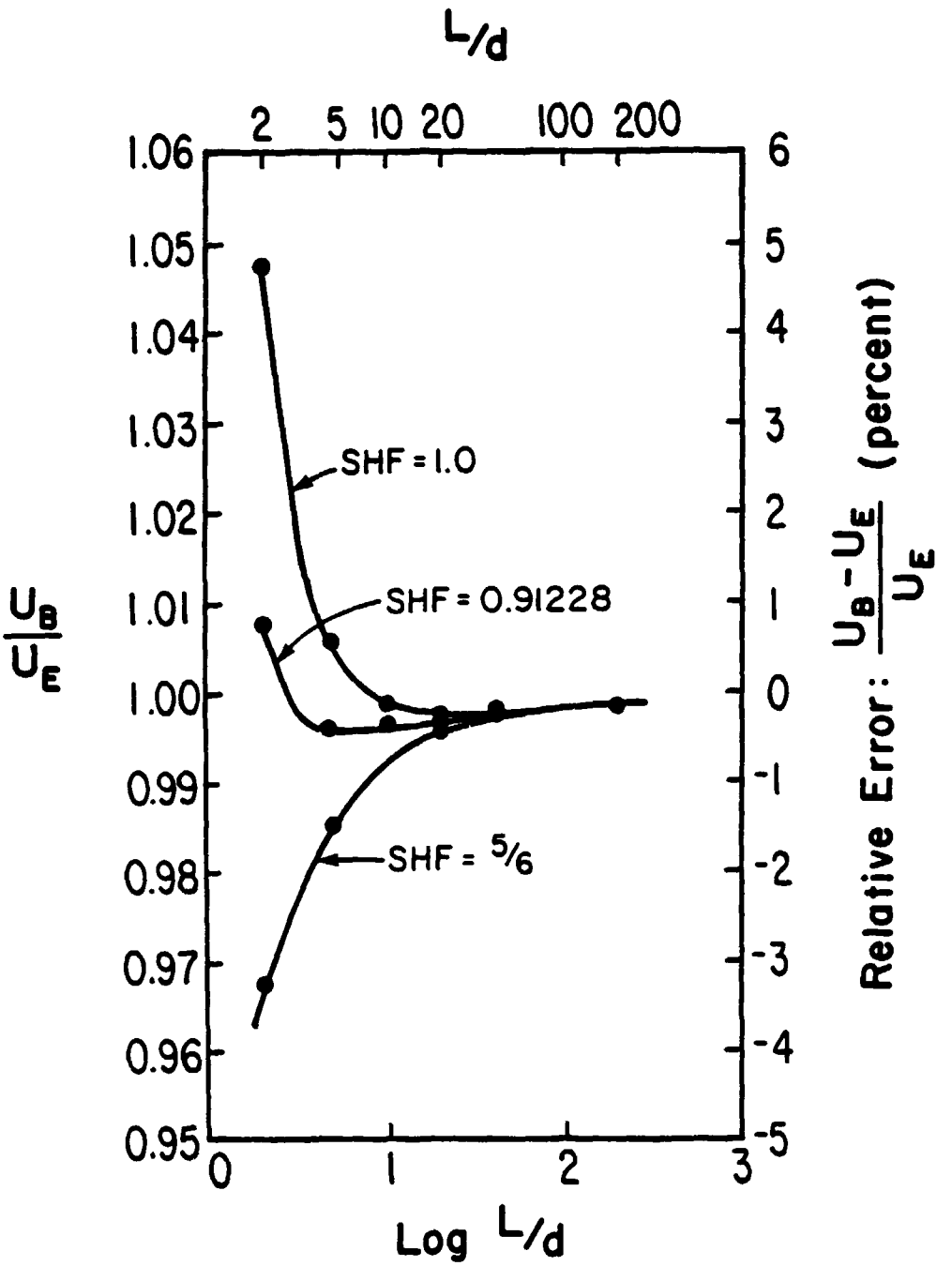


Fig. 5.3.2

Infinite plate strip. Imposed shear displacement. Ratio of strain energies computed from the plate bending model based on the Reissner-Mindlin theory (U_B) and from the theory of elasticity (U_E) vs. length to thickness ratios.

5.4 Simply supported square plate under uniform load

We shall compare the convergence characteristics of the h and p-versions of the finite element method on the basis of the simply supported square plate under uniform load, the solution of which is very smooth. The bases of comparison are: L/d ratios and shear factors.

The rectangular plate domain and the initial element mesh are shown in Fig. 5.4.1

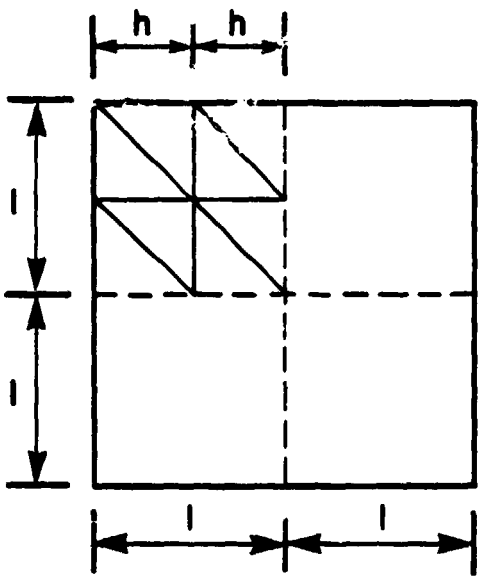


Fig. 5.4.1

Simply supported square plate
Initial finite element mesh

Figures 5.4.2, 5.4.3 and 5.4.4 illustrate the effect of the length to thickness ratio on the rates of convergence. It is seen that as the length to thickness ratio increases from 50 to 1000, the rate of convergence of the p-version based on uniform mesh refinement gradually

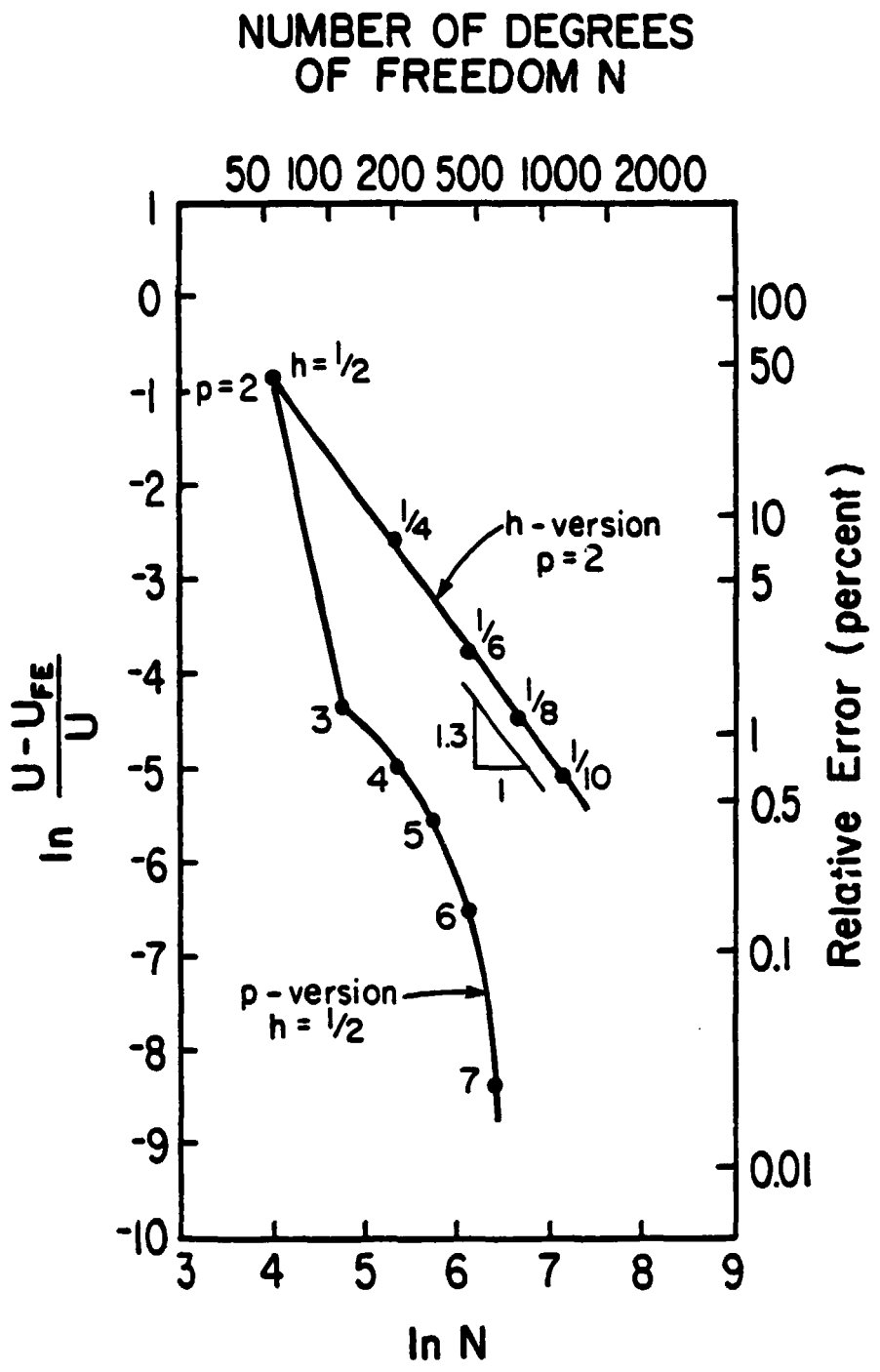


Fig. 5.4.2

Uniformly loaded, simply support square plate. Length to thickness ratio : 50; shear factor: 5/6, Poisson's ratio : 0.3.

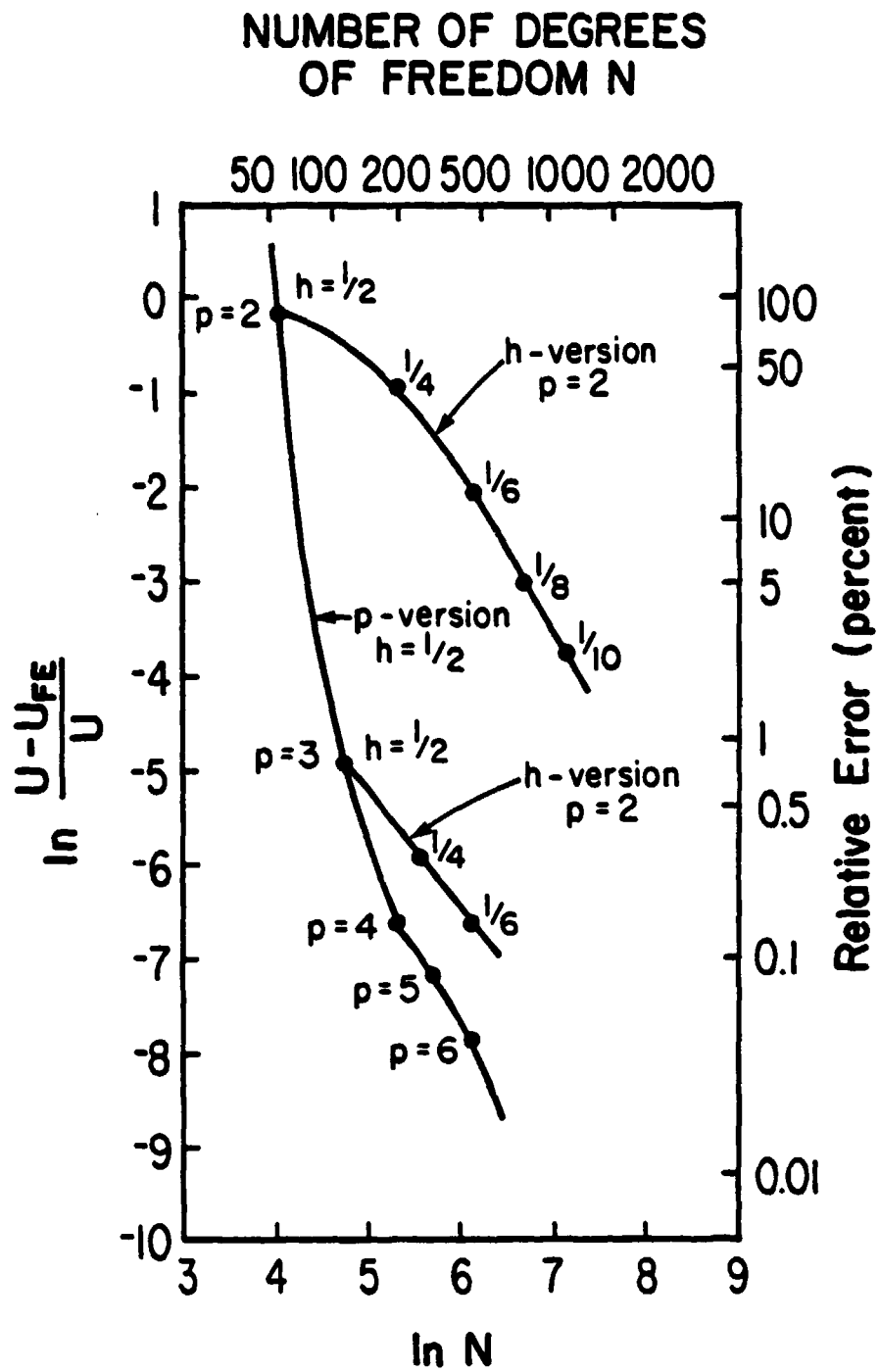


Fig. 5.4.3

Uniformly loaded, simply supported square plate.
Length to thickness ratio : 200
Shear factor : 5/6, Poisson's ratio : 0.3

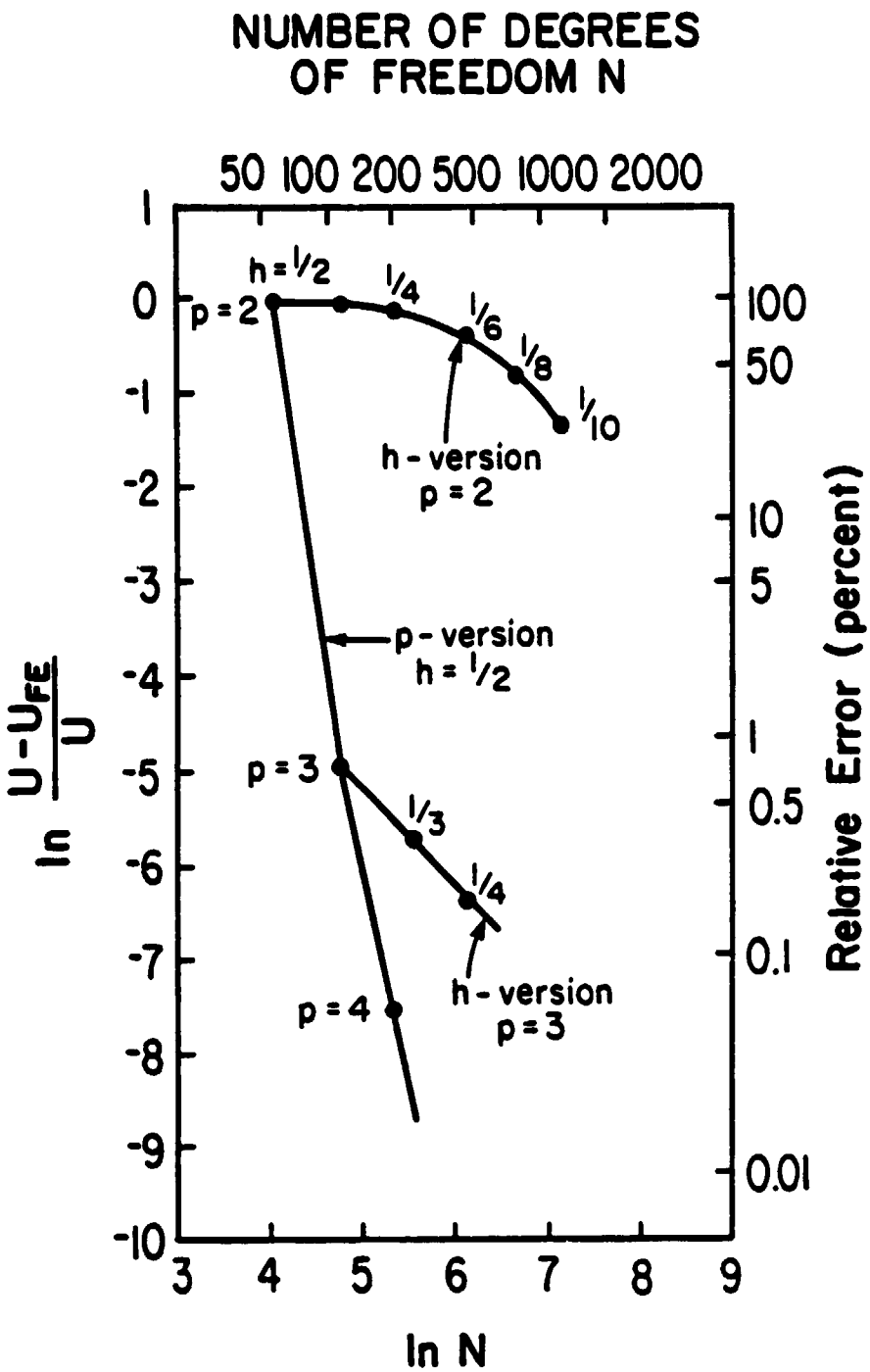


Fig. 5.4.4

Uniformly loaded simply supported square plate
Length to thickness ratio : 1000
Shear factor : 5/6 Poisson's ratio : 0.3

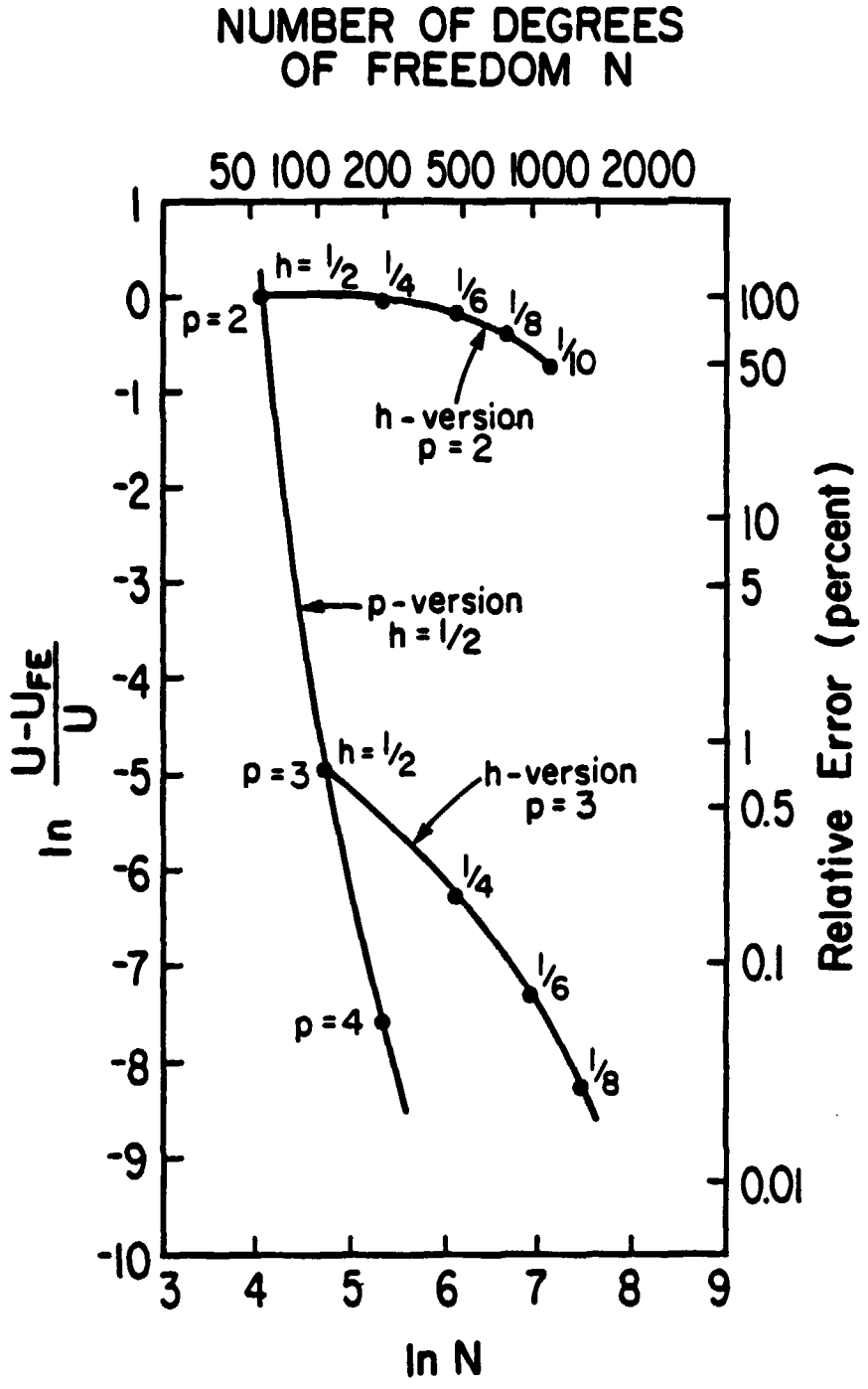


Fig. 5.4.5

Uniformly loaded, simply supported square plate. Length to thickness ratio : 50
Shear factor : 1000 Poisson's ratio : 0.3

diminishes in the preasymptotic range for p = 2. This is similar to the case of nearly incompressible solids discussed in conjunction with the parabolically loaded square panel in Section 3.2. For larger p values the error is already only about one percent, which is due to the fact that this example problem has a very smooth solution. It is seen that for p = 2 the preasymptotic rate of convergence is not significantly affected by length to thickness ratios in either the h or the p-version.

The effect of shear factor is shown in Figures 5.4.2 and 5.4.5. In both cases the length to thickness ratio is 50, however in Fig. 5.4.2 the shear factor is the commonly used value of 5/6 whereas in Fig. 5.4.5 the shear factor is 1000, a value often used to "suppress" shear deformation and obtain a solution which is close to the Kirchhoff plate bending solution. The high value of the shear factor has the same effect as the high length to thickness ratio: the preasymptotic rate of convergence of the h-version slows considerably for p = 2. The reason for the similarity is that in both cases the shear strain energy is multiplied by a penalty term: in one case the penalty term is the length to thickness ratio squared, in the other case it is the shear factor times the length to thickness ratio squared.

5.5 The rhombic plate problem

In the preceding section we examined a problem the solution of which was very smooth and employed uniform mesh refinement. We shall now turn our attention to a much more difficult problem which has received a great deal of attention [24,25,26]. In this case we shall employ a non-quasiuniform mesh refinement which is "natural" for this problem. The problem itself is a simply supported, uniformly loaded

rhombic plate, the acute angles of which are 30 degrees. The plate and the mesh refinement for one quarter of the plate are shown in Fig. 5.5.1.

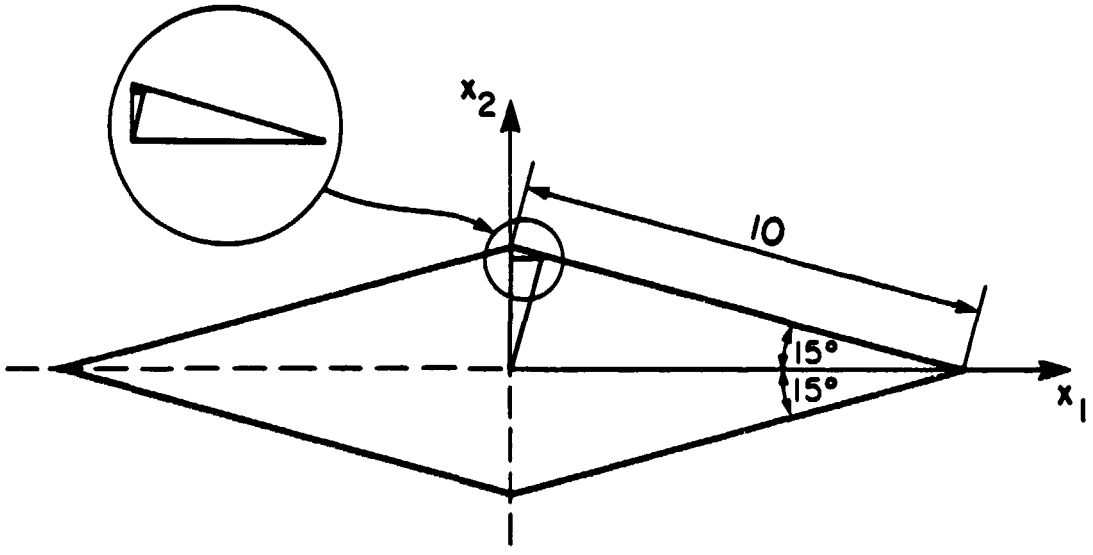


Fig. 5.5.1

The rhombic plate problem
Non-quasiuniform mesh refinement

A very strong singularity occurs at the obtuse vertex when Kirchhoff's theory is applied to this problem. A detailed discussion is available in [26]. As a result, finite element models employing exactly and minimally conforming C^1 finite elements fail to converge within an acceptable range of number of degrees of freedom. For example, Sander reported approximately 17 percent error in the displacement of the centroid when a conforming displacement model was employed with 1000 degrees of freedom [25]. As yet unpublished numerical

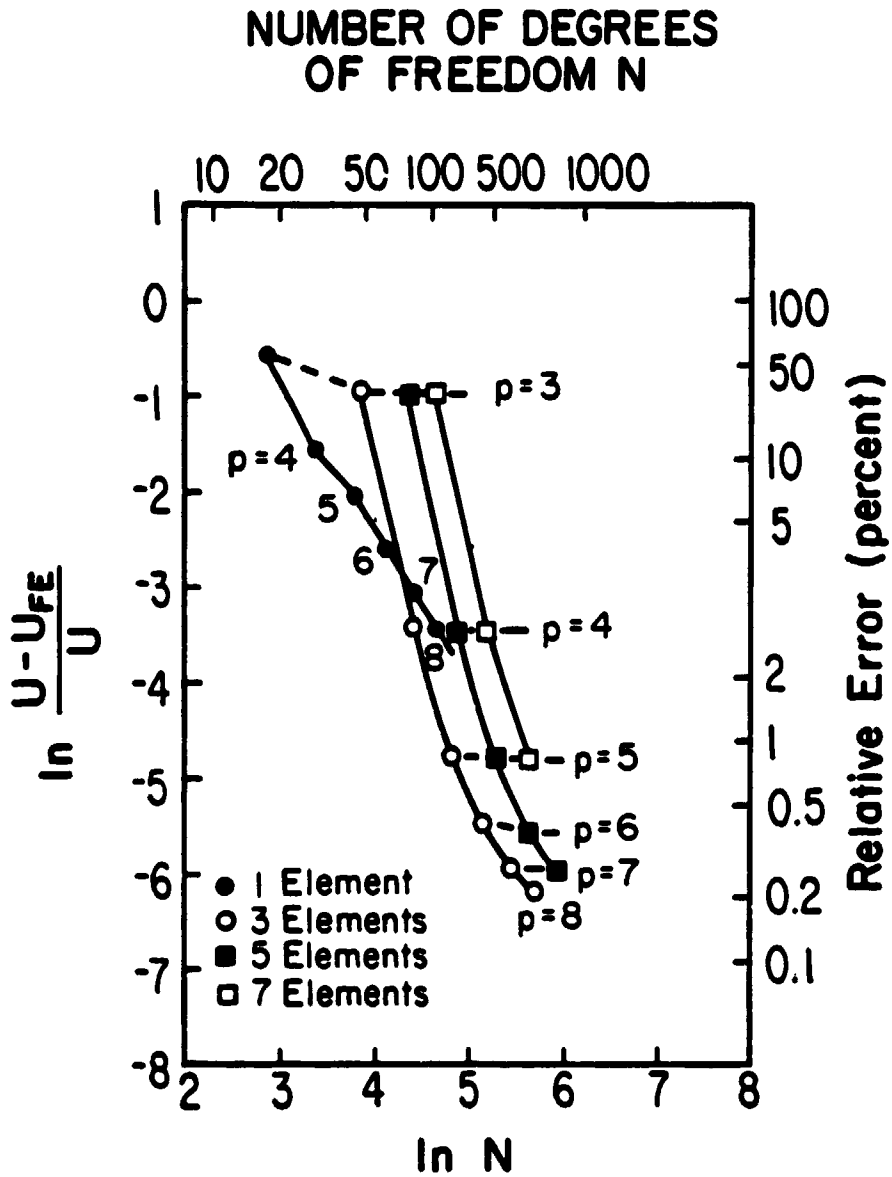


Fig. 5.5.2

Rhombic Plate Problem

Relative error in strain energy vs. Number of degrees of freedom. Non-quasiuniform mesh refinement. Poisson's ratio : 0.3, shear factor : 5/6

studies performed at Washington University also indicated that fifth order exactly and minimally conforming C^1 finite elements converge very slowly even when rational functions are introduced. (Detailed discussion of rational functions is available in [7]). These observations underline the importance of employing the Reissner-Mindlin theory for modeling structural plates. The severe pollution in Kirchhoff plate models caused by obtuse corners is not acceptable in practice.

In this example the finite element mesh is strongly graded toward the obtuse corner. As the number of finite elements takes on the values n = 1, 3, 5, 7... the hypotenuse of the smallest element is $10(1-\cos^2\frac{\pi}{6})^{\frac{n-1}{2}}$. In other words, the size of the smallest element decreases in geometric progression with common factor $(1 - \cos^2\frac{\pi}{6}) \cong 0.07$. Such non-quasiuniform refinement was shown to be optimal when coupled with non-quasiuniform p-distribution [9,27]. In this case the p-distribution is uniform. The result are shown in Fig. 5.5.2. The following observations can be made:

1. If only one element is used and the relative error in energy is to be reduced to under three percent then p must be higher than 8 which is the highest value currently permitted by COMET. (p refers to each of the three fields: one transverse displacement and two rotations).

2. A reasonable grading of elements, such as the 3-element mesh, appears to be the best policy: with p = 5 only, the relative error in energy is under one percent.

3. The 5 and 7 element meshes do not provide significantly greater accuracy than the 3-element mesh. The reason for this is that the error is controlled by the largest element, the size of which is not changed in this non-quasiuniform mesh refinement. Only an adaptive mesh refinement and/or p-distribution scheme is capable of eliminating the wasteful assignment of degrees of freedom which almost invariably occurs when the relative contribution of elements to the total error of approximation is unknown.

This example shows that even in the presence of very severe corner singularities it is possible to obtain approximations, accurate to within one percent, with very few finite elements and low polynomial order (p = 5) using the Reissner-Mindlin theory of plates. The solutions presented here for the 3-element mesh and progressively higher p-values are by far the most efficient of the solutions published to date for this example problem.

6. LOCAL ERROR INDICATORS

Not counting input errors, there are four sources of error in finite element analysis: (1) low polynomial order; (2) singularities due to reentrant corners or sudden changes from one type of support condition to another along a geometrically smooth boundary; (3) singularities due to applied load (e.g. concentrated force) and (4) roundoff errors.

Low polynomial order can be the source of two kinds of error: it may limit the rate of convergence as predicted by eg. (3.1.2) when $\alpha > p$. This is not very important in practical applications because α is generally smaller than p. On the other hand, low p values can cause non-robust behavior, which may be very difficult to detect in practical computations.

Singularities due to reentrant corners are the most common and the least avoidable sources of error. In most practical problems there are several corners. In fact, the more complicated the geometry, the more frequently will corners of the type discussed in Section 3.1 occur. Sudden changes in material or geometric properties also belong in this category.

Imposition of severe singularities due to applied loads is generally avoidable. The use of concentrated forces in elasticity is a convenience but should be avoided when a finite element model based on the displacement formulation is used because the strain energy associated with concentrated forces is unbounded.

Roundoff errors are less of a problem in the p-version than in the h-version. The roundoff error depends on the choice of basis functions and is, therefore, controllable to a certain extent in the development stage of finite element software systems. Full analysis

of the roundoff characteristics of the p-version is not yet available. The results of a preliminary study were presented in [10].

From the foregoing discussion it is evident that in the case of the p-version the main sources of error are corner singularities. The asymptotic rate of convergence is governed by the most severe of the corner singularities. The error depends on the mesh, the polynomial order and the loading as well. The contribution of elements to the total error of approximation generally varies from element to element. In order to minimize the number of degrees of freedom needed for achieving a given level of precision, it is necessary to vary the p-distribution over the finite element mesh so as to make the contribution of each element to the total error approximately the same. In the following we shall be concerned with this problem.

Reliable local indicators for the contribution of individual finite elements to the total error in energy are available for the h-version. These indicators are based on the residuums and jump discontinuities and have the property that the sum of the indicators estimate the total error in energy.

Previous numerical studies have shown that estimators can be constructed for the p-version as well which have approximately the same rate of convergence as the total error in energy but the question of whether the same or similar estimators would serve as local indicators for the relative contribution of individual finite elements to the total error in energy has not been examined.

We have investigated a variant of the estimator given in [28], the functional form of which was proposed by Babuska [29]. First we define our local indicators and demonstrate that the sum of the local indicators

indeed has the same rate of convergence as the error in energy when uniform p-distribution is used on a fixed finite element mesh. We then demonstrate through example problems that the indicators also serve to estimate the relative contribution of individual finite elements to the total error in energy and provide for establishing optimal or nearly optimal p-distribution for a given mesh. Finally, alternative policies for achieving optimal p-distribution in practical computations are discussed.

6.1 Definition of local error indicators

Local error indicators are computed from the residuums and jump discontinuities of the finite element solution. The residuums r_i are obtained from substituting the finite element solution into the equilibrium equation. In elasticity:

$$r_i = G\tilde{u}_{i,jj} + (\lambda + G)\,\tilde{u}_{j,ji} + X_i \qquad (6.1.1)$$

in which r_i is the i^{th} component of the unbalanced body force (residuum); λ and G are the Lame parameters, $\tilde{u}_i$ is the i^{th} component of the displacement vector computed by the finite element method; X_i is the imposed body force.

The jump discontinuities are the differences between the surface tractions computed from the displacement functions belonging to neighboring elements along the common finite element boundary. We shall denote jump discontinuities by $\Delta t_m^{(i)}$ in which i is the edge number, m is the usual component index.

Our goal is to construct an a posteriori error indicator which measures the relative contribution of each finite element to the total error of approximation. The error indicator for the k^{th} element ξ_k consists of two parts; one based on residuums, the other based on jump discontinuities, and has the following form:

$$\xi_k = R_k + C\ T_k \tag{6.1.2}$$

in which R_k is the "R indicator"; C is a constant; T_k is the "T indicator" for the k^{th} element. We shall now briefly outline the rationale for constructing R_k and T_k.

6.1.1 The R indicator

The residuum computed from the finite element solution is proportional to the error in the second derivatives: Suppose that the exact solution u_i is known. Then:

$$0 = G\ u_{i,jj} + (\lambda + G)\ u_{j,ji} + X_i \tag{6.1.3}$$

and, from equations 6.1.3 and 6.1.1:

$$r_i = G\ (u_{i,jj} - \tilde{u}_{i,jj}) + (\lambda + G)\ (u_{j,ji} - \tilde{u}_{j,ji}) \tag{6.1.4}$$

We seek to approximate the error in strain energy which is the integral over the finite element domain of a linear combination of the squares of the error in the first derivatives. In view of this fact, the R estimator is constructed as follows:

$$R_k = \frac{A_k}{E\ p_k^2} \int_{A_k} \delta_{ij}\ r_i\ r_j\ W_k\ dA + \bar{R}_k + \hat{R}_k \qquad (6.1.5)$$

in which

A_k is the area of the k^{th} element;

E is the modulus of elasticity;

p_k is the polynomial order of the displacement approximation over the k^{th} element;

r_i is the unbalanced body force vector component, defined in eq. 6.1.1

δ_{ij} is the Kronecker delta;

W_k is a weight function, defined as the square root of the first internal mode; i.e. the square root of a cubic polynomial that vanishes along the boundaries of the k^{th} element. W_k is normalized such that:

$$\frac{1}{A_k} \int_{A_k} W_k\ dA = 1 \qquad (6.1.6)$$

$\bar{R}_k$ and $\hat{R}_k$ are correction terms which are zero when the p-distribution over the k^{th} element and its immediate neighbors is uniform. When this condition is not met then the k^{th} element has higher and/or lower order neighbors than p_k. The lower order neighbors downgrade the k^{th} element along their common boundaries. Similarly, the k^{th} element downgrades its higher order neighbors. An indicator of the relative contribution of the k^{th} element to the total error in energy must in some way account for these effects. $\bar{R}_k$ and $\hat{R}_k$ were devised for that purpose on the basis of extensive numerical experiments. We shall now define these correction terms:

Let I_k be the set of element numbers which have common edge with element k. Also, let

$$\bar{p}_k = \min_{i \in I_k} \{p_i\} \tag{6.1.7}$$

and

$$\bar{I}_k = \{i \mid p_i < p_k \,,\, i \in I_k$$

If $\bar{p}_k \geq p_k$ then $\bar{R}_k = 0$. Otherwise:

$$\bar{R}_k = \left[1 + p_k^2 \sum_{i \in \bar{I}_k} \frac{1}{p_i^2}\right]^{-1} \left[\frac{1}{\bar{p}_k^2} - \frac{1}{p_k^2}\right] \frac{A_k}{E} \int_{A_k} \delta_{mn}\, r_m\, r_n\, W_k \, dA \tag{6.1.8}$$

Similarly, let $\hat{p}_k = \max_{i \in I_k} \{p_i\}$ and

$I_k = \{i \mid p_i > p_k \,,\, i \in I_k\}$. If $\hat{p}_k \leq p_k$ then $\hat{R}_k = 0$. Otherwise:

$$\hat{R}_k = \sum_{i \in \hat{I}_k} \left[\frac{p_k^2}{p_i^2} + \sum_{j \in \bar{I}_i} \frac{p_k^2}{p_j^2}\right]^{-1} \left[\frac{1}{\bar{p}_i^2} - \frac{1}{p_i^2}\right] \frac{A_i}{E} \int_{A_i} \delta_{mn}\, r_m\, r_n\, W_k \, dA \tag{6.1.9}$$

$\bar{R}_k$ is the correction term that accounts for the downgrading of the k^{th} element by its neighbors; $\hat{R}_k$ accounts for the k^{th} element downgrading its neighbors.

Further details are available in reference 30.

6.1.2 The T indicator

The T-indicator accounts for jump discontinuities at interelement boundaries and the error between computed and specified tractions at external boundaries:

Let I_k be the set of edges of element k for which surface tractions are specified and let $I_k^* = I_k \cup I_k$.

$$T_k = \sum_{i \varepsilon I_k^*} \frac{\ell_i}{EP_{ik}} \int_{\ell_i} \delta_{mn} \Delta t_m^{(i)} \Delta t_n^{(i)} w_k^{(i)} \, d\ell \qquad (6.1.10)$$

in which

ℓ_i is the length of the i^{th} edge of element k;

$P_{ik} = p_k + p_i$ when $i \varepsilon I_k$;

$P_{ik} = 2\, p_k$ when $i \varepsilon \tilde{I}_k$ and no displacement vector component is prescribed along edge i;

$P_{ik} = p_k$ when $i \varepsilon \tilde{I}_k$ and one displacement vector component is prescribed along edge i;

$\Delta t_m^{(i)}$ is the difference between the surface tractions calculated from the finite element solutions for element k and element i along their common edges when $i \varepsilon I_k$. When $i \varepsilon \tilde{I}_k$ then $\Delta t_m^{(i)}$ is the difference between the surface tractions calculated for element k and the prescribed surface tractions along edge i;

$w_k^{(i)}$ is the square root of the first internal edge mode for edge i, normalized such that

$$\frac{1}{\ell_i} \int_{\ell_i} w_k^{(i)} \, d\ell = 1$$

Finally, we define the error indicator for the k^{th} element ξ_k as:

$$\xi_k = R_m + c\, T_k \tag{6.1.11}$$

in which $c = \frac{\Sigma R_k}{\Sigma T_k}$ with summation over all elements.

6.2 Example: Parabolically loaded square panel

This problem is the same as the problem discussed in Section 3.2, however the finite element mesh is different (see Fig. 6.2.1).

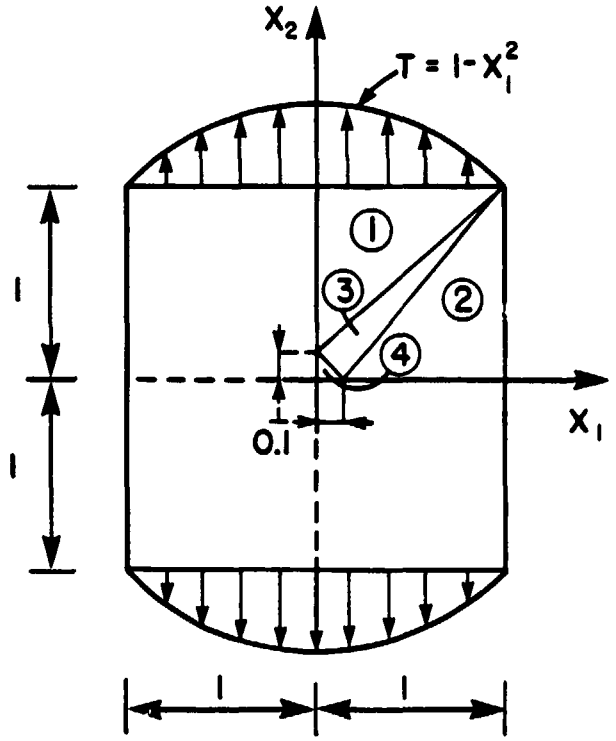


Fig. 6.2.1

Parabolically loaded square panel
Finite element mesh and element numbers
Poisson's ratio : 0.3

We first demonstrate that ΣR_k and ΣT_k (summation over all elements) have the same rate of convergence as the total error in energy when uniform p-distribution is used on a fixed finite element mesh. In Fig. 6.2.2 we have plotted the error in strain energy (i.e. the error in energy norm squared $||e||_E^2$); ΣR_k and ΣT_k on log scale against N^{-1}, also on log scale. It is seen that the slopes of the three curves are very nearly indentical, especially for large N, indicating that the asymptotic rates of convergence are also nearly identical.

Next we demonstrate that ξ_k is a reliable indicator of the relative contribution of element k to the total error in strain energy: Starting from uniform p = 2, we incremented p, one element at a time by one, for all elements, and computed the corresponding errors in energy. We then selected that p-distribution for which the error was the smallest and using this p-distribution as the base we again incremented p by one, one element at a time, for each element and selected that p-distribution for the next base for which the error was the smallest. We continued this process of exhaustive evaluation until p had to be incremented beyond 8 which is the limiting value for our computer program, COMET-X. In this way we obtained an optimal sequence of p-distributions. Next, starting again from uniform p = 2, we computed ξ_k for all elements and incremented p by one in that element only for which ξ_k was maximum. Proceeding in this way we obtained an indicated sequence of p-distributions. Both the optimal and indicated sequences are shown in Table 6.2.1. The two sequences are not identical only in the third and fourth cycles. For these cycles the results of the exhaustive evaluation are tabulated. Under "Remarks" the indicated and optimal sequences are noted. The

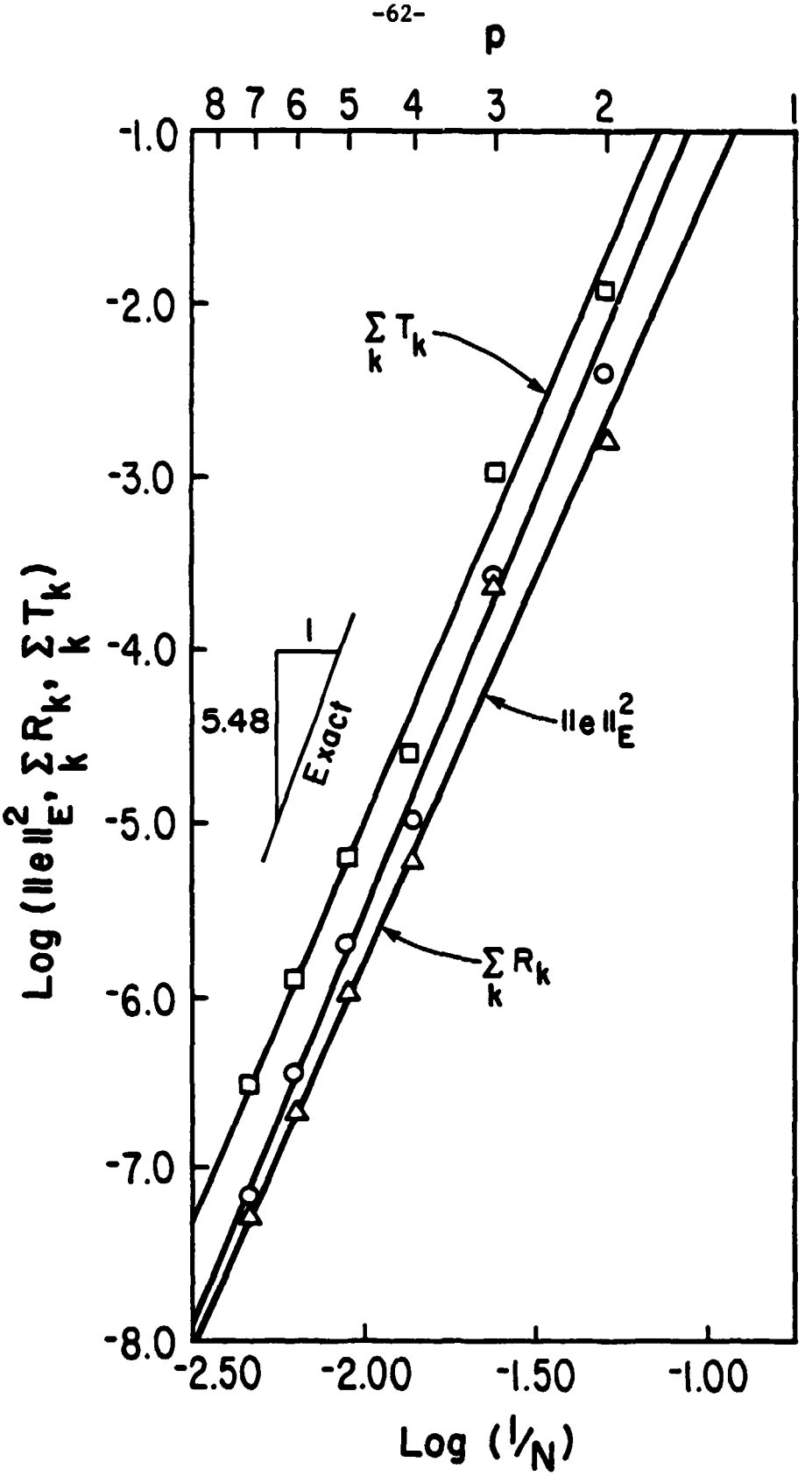


Fig. 6.2.2

Parabolically loaded square panel
Error in strain energy and sums of the R and T indicators over all finite elements

Table 6.2.1

Optimal and Indicated Sequences of P-Distribution for the Parabolically Loaded Square Panel

P	N	$100\,\frac{U-U_{FE}}{U}$	ρ	Remarks
2 2 2 2	20	1.70	0.53	
3 2 2 2	25	6.68 E-1	0.48	
4 2 2 2	32	5.38 E-1	0.90	Optimal
3 3 2 2	30	5.66 E-1	0.80	
3 2 3 2	29	6.51 E-1	0.79	Indicated
3 2 2 3	29	6.68 E-1	0.80	
4 2 3 2	36	5.16 E-1	1.12	Indicated
3 3 3 2	36	1.13 E-1	0.64	Optimal
3 2 4 2	33	6.50 E-1	1.02	
3 2 3 3	35	6.51 E-1	1.15	
4 3 3 2	43	6.15 E-2	0.74	
4 4 3 2	50	1.63 E-2	0.61	
4 4 4 2	58	4.34 E-3	0.51	
5 4 4 2	67	2.94 E-3	0.59	

Note: The polynomial orders are listed in the order of element numbering shown in Fig. 6.2.1.

Table 6.2.2

Sequence of P-Distributions Obtained on the Basis of Elemental Errors in Strain Energy for the Parabolically Loaded Square Panel

P	N	$100\,\frac{U-U_{FE}}{U}$	ρ
2 2 2 2	20	1.70	0.53
3 2 2 2	25	6.68 E-1	0.48
4 2 2 2	32	5.38 E-1	0.90
5 2 2 2	41	5.29 E-1	1.47
6 2 2 2	52	5.29 E-1	2.36
7 2 2 2	65	5.28 E-1	3.68
8 2 2 2	80	5.28 E-1	5.58

See footnote to Table 6.2.1.

number of degrees of freedom, the relative error in energy (percent) and a cost ratio ρ, defined as the approximate cost associated with the non-uniform p-distribution divided by the corresponding cost for uniform p-distribution for the same relative error in energy (assuming asymptotic behavior) are also shown in Table 6.2.1.

Occasional differences between the indicated and optimal sequences can be explained by pointing out that the effect of downgrading one element by its neighbors cannot be estimated with precision. It is important only that the indicators be stable in the sense that if deviations occur from the optimal path then, following the indicators, it should be possible to return to the optimal path. We note also that small deviations are unimportant in practical computations where p would be incremented concurrently for groups of elements for which the indicators are within a certain tolerance range.

Finally, we make the following observation: The indicators are not to be confused with indications of error in strain energy for individual finite elements. In fact, the error in strain energy for individual finite elements is a very poor indicator. To illustrate this, starting from uniform $p = 2$ we incremented p by one at a time for that finite element only for which the error in strain energy was the largest. The resulting sequence of p-distribution is shown in Table 6.2.2. The sequence is rapidly deviating from the optimal one which can be judged from the progressively increasing value of ρ.

6.3 Example: Double-edge cracked square panel

Our next example is the double-edge cracked square panel, shown in Fig. 6.3.1. The problem is essentially the same as the problem discussed

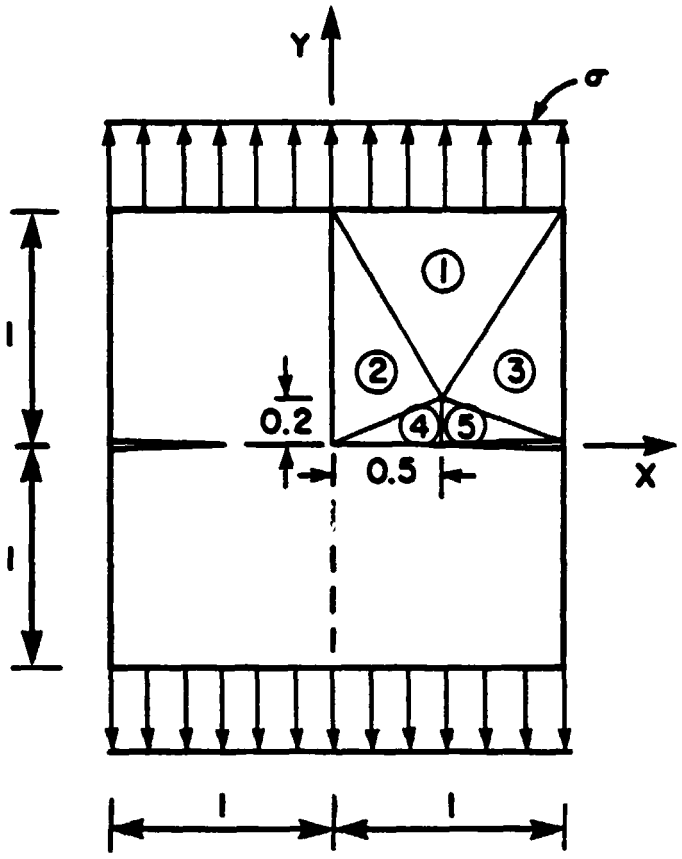


Fig. 6.3.1

Double-edge cracked square panel
Finite element mesh and element numbers
Poisson's ratio : 0.3

under section 4.3, but the finite element mesh is different. As in the preceding example, we obtained an optimal sequence of p-distributions by exhaustive evaluation, however in this case we started from p = 3 (uniformly). Also, we obtained an indicated sequence of p-distributions using the indicator ξ_k. The results are given in Table 6.3.1. Remarkably, the indicated sequence deviates from the optimal one only in the seventh and eighth cycles. The relative error for the optimal and indicated sequences is shown in Table 6.2.1 and plotted in Fig. 6.3.2.

Table 6.3.1

Optimal and Indicated Sequences of P-Distribution for the Edge Cracked Square Panel

p	N	$100\,\frac{U-U_{FE}}{U}$	ρ	Remarks
3 3 3 3 3	54	4.61	1.01	
3 3 3 3 4	60	3.58	0.75	
3 3 3 3 5	68	3.06	0.71	
3 3 3 4 5	75	2.56	0.60	
3 3 3 4 6	85	2.23	0.59	
3 3 3 5 6	94	1.94	0.55	
4 3 3 5 6	100	1.85	0.56	
3 4 3 5 6	101	1.85	0.57	
3 3 4 5 6	102	1.74	0.51	Optimal
3 3 3 6 6	105	1.81	0.59	
3 3 3 5 7	106	1.74	0.55	Indicated
4 3 3 5 7	112	1.62	0.54	
3 4 3 5 7	113	1.62	0.55	
3 3 4 5 7	114	1.52	0.49	Optimal
3 3 3 6 7	117	1.59	0.56	Indicated
3 3 3 5 8	120	1.61	0.61	
3 3 4 6 7	125	1.38	0.48	
3 3 4 6 8	139	1.24	0.48	

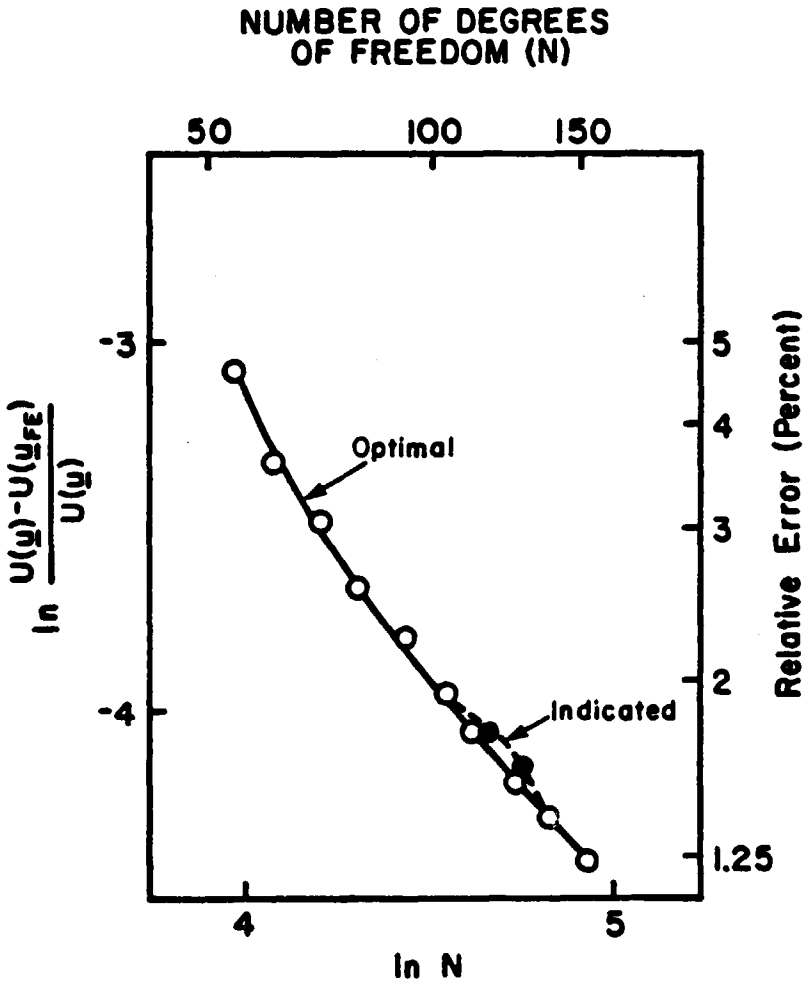


Fig. 6.3.2

Double-edge cracked panel
Relative error vs. number of degrees of freedom for the optimal and indicated sequences of p-distribution

This example shows that the error indicator is a reliable guide even when a strong singularity is present. It is also seen that the cost savings achievable with optimal p-distribution, as compared with uniform p-distribution, are between 40 and 50 percent (the value of 1-ρ expressed as percentage). These savings are strongly mesh dependent, however. For finer meshes the savings would be greater; for coarser meshes the savings would be smaller.

6.4 Example: Lap joint problem

In this problem we have a fairly complicated geometry and 10 geometric singularities of various intensity (Fig. 6.4.1). The plates are joined

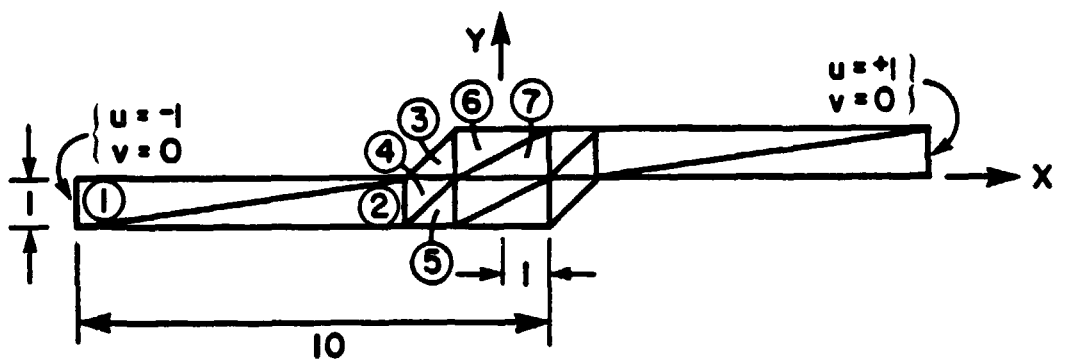


Fig. 6.4.1

Welded lap-joint
Finite element mesh and element numbering
Poisson's ratio : 0.3

Table 6.4.1

Optimal and Indicated Sequences of P-Distribution
for the Lap-Joint Problem

p	N	$100\,\frac{U-U_{FE}}{U}$	ρ	Remarks
2 2 2 2 2 2 2	72	29.539	21.22	
3 2 2 2 2 2 2	80	25.935	20.20	
3 3 2 2 2 2 2	92	4.418	0.77	
4 3 2 2 2 2 2	104	3.688	0.69	
4 3 3 2 2 2 2 4 4 2 2 2 2 2	112 120	3.244 3.435	0.62 0.80	Optimal Indicated
4 3 3 3 2 2 2 5 4 2 2 2 2 2	124 136	2.896 3.128	0.60 0.85	Optimal Indicated
5 3 3 3 2 2 2 5 4 2 3 2 2 2	140 144	2.579 2.832	0.61 0.78	Optimal Indicated
5 4 3 3 2 2 2 5 5 2 3 2 2 2	156 164	2.283 2.536	0.59 0.81	Optimal Indicated
5 5 3 3 2 2 2	176	2.050	0.61	
5 5 3 3 3 2 2 5 6 3 3 2 2 2	188 196	1.850 1.955	0.57 0.69	Optimal Indicated
6 5 3 3 3 2 2 5 6 3 3 2 2 3	208 204	1.691 1.839	0.58 0.66	Optimal Indicated
6 5 4 3 3 2 2 6 6 3 3 2 2 3	220 228	1.554 1.638	0.54 0.65	Optimal Indicated
6 6 4 3 3 2 2 6 7 3 3 2 2 3	244 252	1.416 1.554	0.56 0.72	Optimal Indicated
6 6 4 3 3 2 3 7 7 3 3 2 2 3	256 280	1.321 1.469	0.54 0.79	Optimal Indicated
6 6 4 3 3 3 3 7 7 3 4 2 2 3	272 292	1.162 1.448	0.47 0.84	Optimal Indicated
7 6 4 3 3 3 3 7 7 3 4 3 2 3	296 308	1.088 1.268	0.49 0.72	Optimal Indicated
7 7 4 3 3 3 3 7 7 4 4 3 2 3	324 324	1.004 1.118	0.50 0.62	Optimal Indicated
7 7 4 3 3 3 4 7 7 4 4 3 3 3	336 340	0.941 0.951	0.47 0.49	Optimal Indicated
7 7 4 4 3 3 4	352	0.898	0.47	

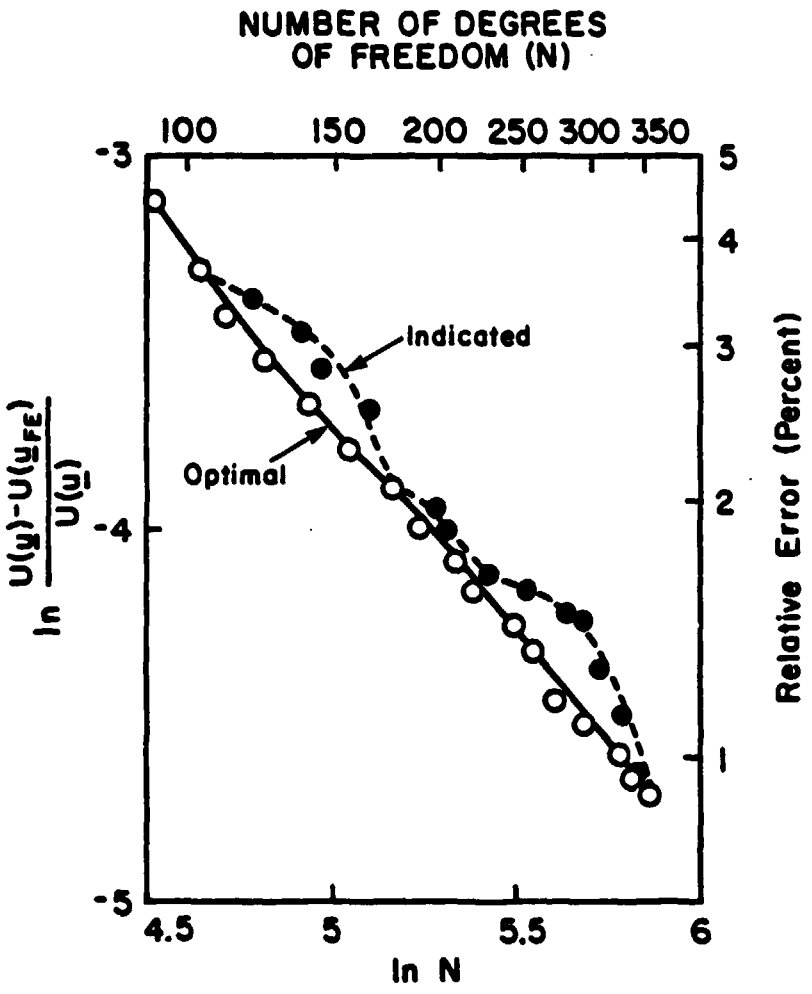


Fig. 6.4.2

Welded lap-joint
Relative error vs. number of degrees of freedom
for the optimal and indicated sequences
of p-distribution

only by the first welds (element 3) and are free at the base of element 7. Loading is by imposed displacement, as shown in Fig. 6.4.1.

The sequences of p-distribution (arranged in the order of element numbering) obtained from exhaustive evaluation and from evaluations based on the error indicator are shown in the first column of Table 6.4.1. The relative error is plotted against the number of degrees of freedom for both sequences in Fig. 6.4.2.

The results show that the indicated sequence does not diverge from the optimal sequence. As previously noted, local deviations are attributable to the fact that the effect of downgrading of one element by its neighbors cannot be estimated with precision: depending on the approximated function, the suppressed degrees of freedom may be very important or not important at all from the point of view of contribution to the total error of approximation. Numerical experiments have indicated that deviations are less apt to occur when p is increased in more than one element at a time: for example p could be increased for all elements simultaneously for which the error indicator is greater than the average value of all error indicators.

7. STRUCTURAL SYNTHESIS

In the preceding sections we focused on the reliability and efficiency of the analysis part of the programming system for structural synthesis. We have shown that the desired level of precision closely determines the required number of degrees of freedom, hence the cost of analysis, which is the dominant part of the structural synthesis process from the point of view of resource requirements.

The phrase: "desired level of precision" means two things: the numerical value of the acceptable relative error and definition of the norm in which the error is to be measured. In our analysis the error was measured in energy (the square of the energy norm in use in mathematical works). In a certain sense this is the logical choice because the finite element method based on the displacement formulations actually minimizes the error in energy norm. Other norms, often regarded as more convenient for engineers, do not always exist: For example, in the case of the lap-joint problem, discussed in Section 6.3, there are 10 geometric singularities, 8 of which result in infinitely high stresses. In this case it would not be meaningful to state the desired level of precision in terms of stresses: The error is infinitely large at the singular points. The stresses are approximated in the least squares sense, rather than pointwise, in the displacement formulation of the finite element method.

In structural synthesis the constraints may be stated in a variety of different norms: stress resultants (moments, axial and shear forces) at specific points, displacements, vibration frequencies, margins of safety against buckling, etc. In general, the corresponding acceptable tolerance levels can be given on the basis of engineering considerations.

In order to minimize the cost of analysis it is necessary to select the number and distribution of the degrees of freedom through proper choices of the mesh and polynomial orders so that the acceptable tolerances are neither exceeded nor satisfied with excessive margins.

Ideally, one would need reliable local a posteriori error estimators in the various norms to achieve this goal. At present such estimators are not available. Some preliminary work has been completed under the present project for the p-version in energy norm only, which was reported in Section 6, but much fundamental work remains. The state of the art is more advanced for the h-version [28] but the computer implementation is more difficult. The benefits to be gained from such estimators are twofold: increased confidence in the computed results and reduced cost. The cost reduction will be problem dependent, however it can be estimated to be about 50 percent for most practical problems, based on what is already available through application of the p-version of the finite element method. Reliable local a posteriori estimators in other norms useful in engineering analysis can be developed in two to five years. Full utilization of adaptivity in structural synthesis requires the analyzer to communicate to the solver the required tolerance levels at the beginning of each iterative cycle.

Even in the absence of fully adaptive selection of the proper p-distribution, the p-version provides for very substantial cost reductions in the analysis part of the structural synthesis process. The basis for cost reduction is the faster rate of convergence of the p-version which can be realized almost invariably in engineering analysis. (Wave propagation problems in which the wave front is not at finite

element boundaries are exceptions: the asymptotic rates of convergence of the h and p-versions in such cases are identical [9,10]).

The rationale for utilizing indirect error estimation techniques in conjunction with the p-version of the finite element method and the recommended procedure is as follows: It is known from theoretical analyses that the rate of convergence (i.e. the relative error vs. the number of degrees of freedom relationship) is governed by geometric and loading singularities. This was discussed in Section 3 of this report. Additional details are available in references 9 and 10. The various types of singularities can be readily classified on the basis of the smoothness parameter of the approximated function as it was done by Williams [12] for plane elastic and plate problems. A series of benchmark problems, characterized by various smoothness parameters, can now be used for establishing the relationship between relative error and polynomial order. In a given problem, where several different singularities occur, the conservative approach is recommended: the polynomial order should be selected on the basis of the strongest singularity. For example, referring to figure 4.3.2 we see that p=5 corresponds to a 2 percent relative error for the edge cracked panel problem using a coarse mesh. In fact, if only a 3-element mesh were used the relative error would still be about two percent for p=5 [31].

The smoothness parameter, which measures the strength of the singularity, (the value of α in eq. 3.1.1) is 0.5 in this case. Similarly, the strongest singularity in the welded lap-joint problem has the same smoothness parameter: $\alpha = 0.5$. Thus, if we wish to have a

solution which is accurate to approximately 2 percent in energy for the welded lap-joint problem, we should employ p = 5 uniformly.

7.1 Example: Simply supported rectangular plate with moveable supports.

In this example we compare the computational costs of optimizing a simple problem via the h and p-versions of the finite element method requiring similar levels of precision for both versions and employing existing technology.

The problem definition is given in Fig. 7.1.1. The plate is simply supported, the moveable support is also a simple support. Because of symmetry, only one quarter of the plate needs to be modeled. The boundary conditions and the four-element mesh are shown in fig. (b), the 6 and 96-element meshes are shown in figures (c) and (d). In order to perserve uniformity, in the subsequent discussions the plate thickness is 1.0 in; Poisson's ratio is 0.2; the modulus of elasticity is 30,000 ksi, the shear factor is 5/6; the lateral load is uniform pressure of 5.0 ksi and the position of the intermediate support (x) is 6.0 inches, unless otherwise noted.

We note that there are two singularities, one at point A the other at point B in Fig. 7.1.1(a). It is possible to compute the strength of these singularities by analytic methods, but not necessary. It is sufficient to analyze a model problem with the same singularities, to establish the minimum p-level and minimum mesh refinement needed for the finite element solution.

Because this is a simple problem, we use the problem itself to find the minimal p-level and mesh refinement. This is done in

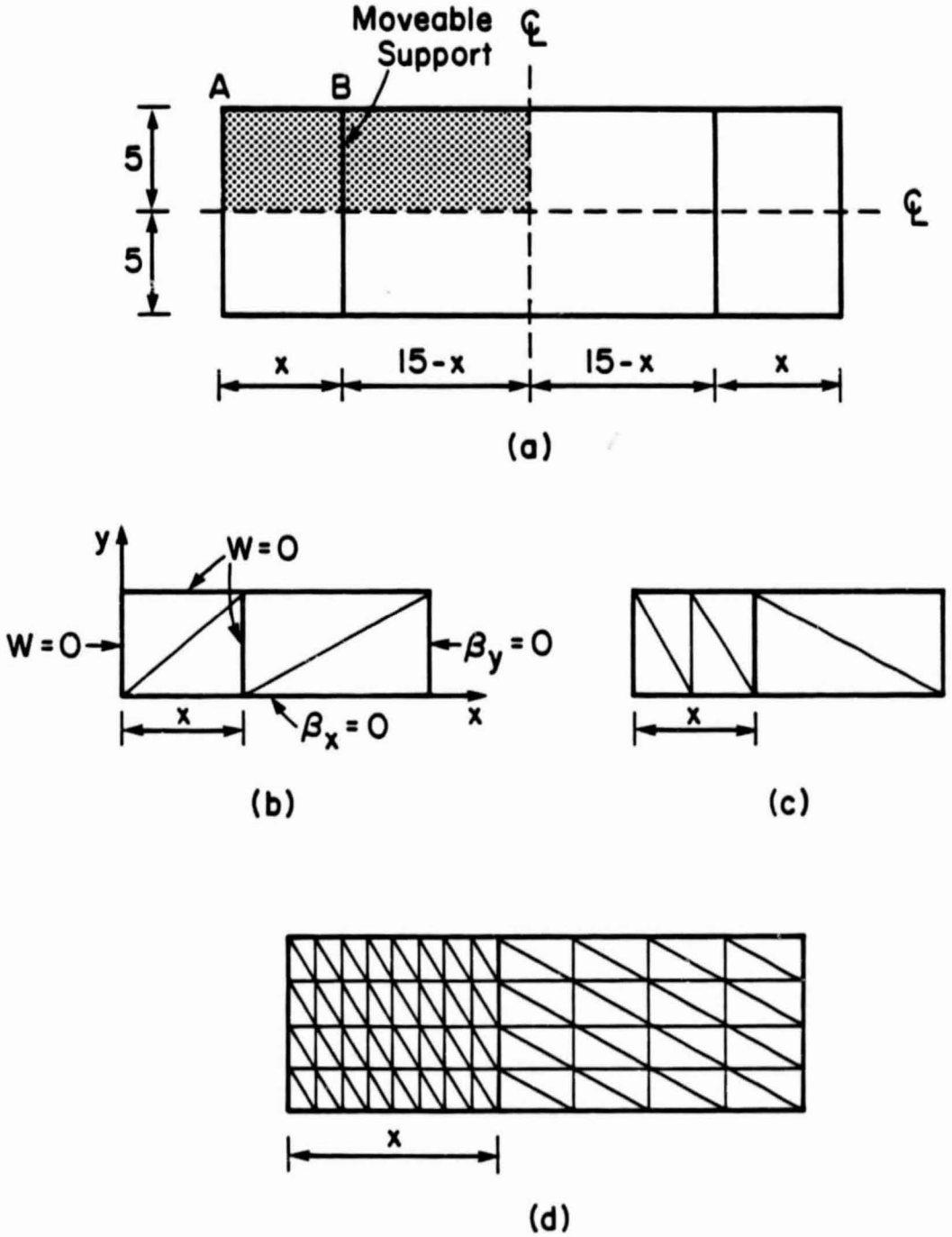


Fig. 7.1.1

Simply supported rectangular plate with moveable internal supports. (a) Dimensions; (b) 4-element mesh and principal boundary conditions; (c) 6-element mesh; (d) 96-element mesh.

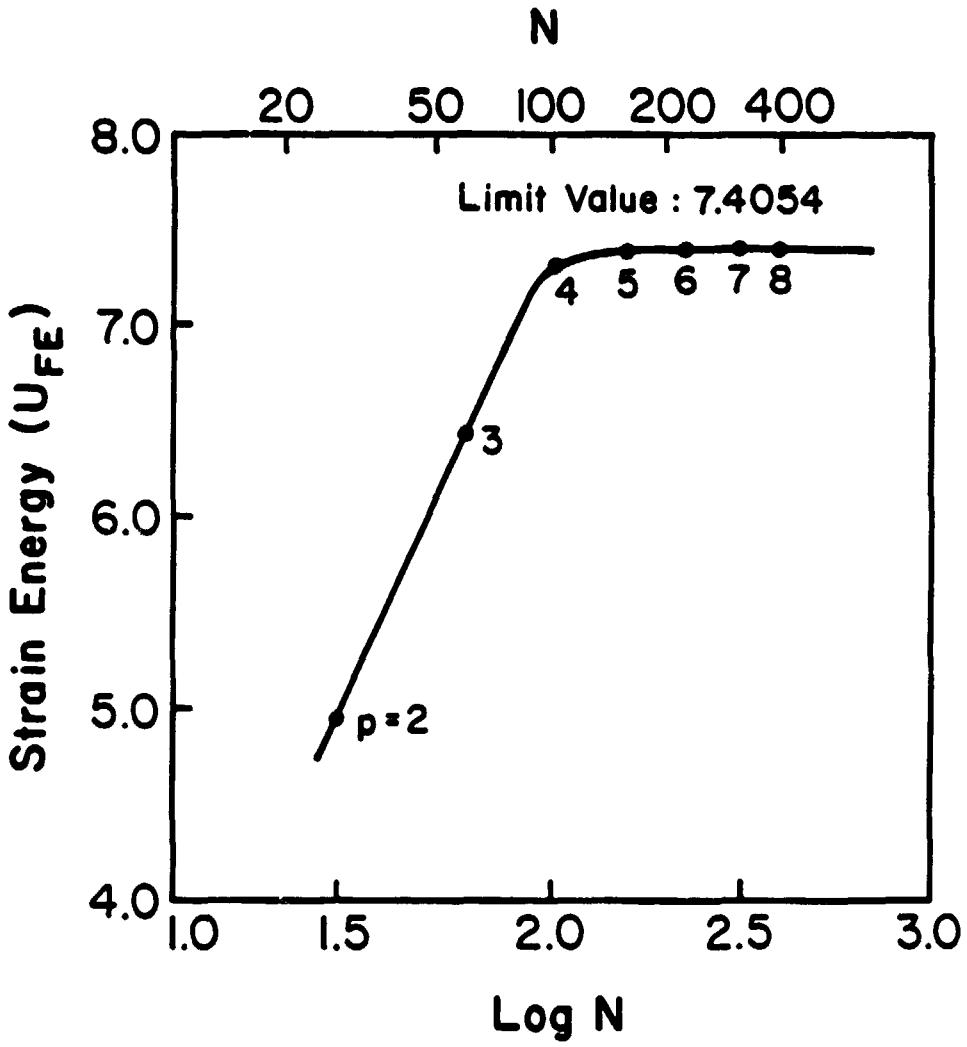


Fig. 7.1.2

Simply supported rectangular plate with moveable supports. Four Element Mesh. Strain energy vs. N. Support location : x = 6.0

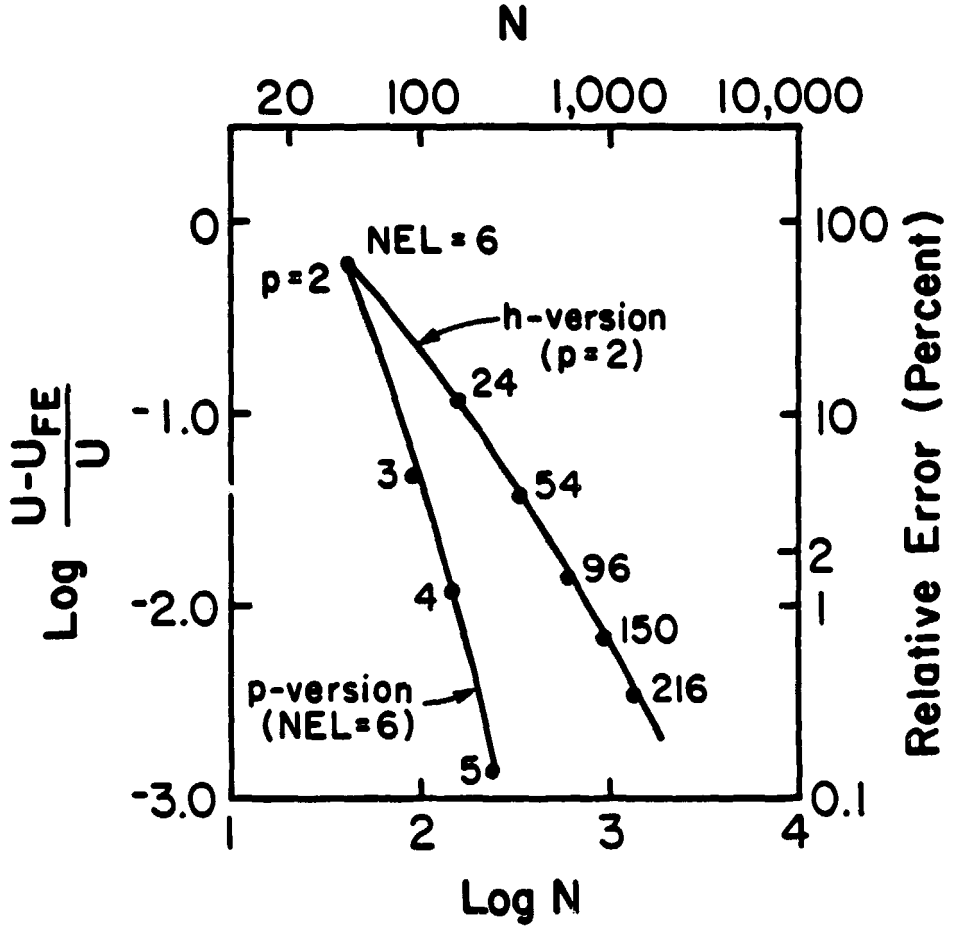


Fig. 7.1.3

Simply supported rectangular plate with moveable supports. Relative error in strain energy vs. N
Support location : x = 6.0

two steps: First we compute the approximate value of the true strain energy, using the 4-element mesh in conjunction with the p-version of the finite element method. The results are shown in Fig. 7.1.2. As p is increased, the value of the strain energy approaches 7.4054 in-k for the quarter plate. In order to illustrate the fact that this result is not sensitive to the mesh, we did a similar analysis for the 6-element mesh and obtained the same result. Using 7.4054 in-k as the estimated true strain energy, we next computed the two extensions (by the h and p versions) of an initial solution in the relative error vs. number of degrees of freedom diagram, with logarithmic scales (Fig. 7.1.3). In this case our initial solution was the 6-element mesh and p = 2. For the h-version p was 2 for all three fields and the mesh was refined quasi-uniformly, as shown in Fig. 7.1.1(d). The refinement allows for varying the position of the support without introducing excessive distortion in the elements. Of course, any h-version finite element code could have been used for the h-extension. In the interest of preserving uniformity we employed COMET for both the h and p-versions.

It is seen from Fig. 7.1.3 that the 96-element mesh with p = 2 and the 6-element mesh with p = 4 yield about the same relative error in strain energy which is between one and two percent. This level of accuracy is probably more than adequate for most engineering purposes. The key computational parameters are listed in Table 7.1.1.

Table 7.1.1

Comparison of solutions by the h and p versions
on the basis of similar accuracies

	h-version	p-version
No. of elements	96	6
Polynomial order (p)	2	4
Relative error in strain energy (percent)	1.45	1.19
No. of degrees of freedom	600	156
Total solution time (CPU sec.)	251	41.5
Solution of equations (CPU sec.)	61	14.5
Computation of stiffness matrices and load vectors (CPU sec.)	125	15.3

The computations were performed on a DEC System 2040 computer; 128K 36-bit work memory, TOPS-20 Operating System. Because the indicated CPU time varies with the overall workload of the machine, exact duplication of the CPU times shown in Table 7.1.1 is not possible. Variations can be as high as 10 percent.

The results are typical in the sense that the total solution time for the h version is about 6 times that for the p-version. In our structural synthesis studies we varied the position of the support (x) and the plate thickness (t). The constraint was that the maximum

stress, computed from the moments in the usual way, should not exceed 20 ksi. For both the h and p versions convergence occurred after 10 or 11 iterations. Thus, using the h-version, the solver required approximately 42 CPU minutes whereas using the p-version the solver required approximately 7 CPU minutes. The results of Section 6 indicate that with the use of optimal p-distribution additional savings of 50 percent should be possible.

In extrapolating these results for other problems, the following points must be considered: (1) The CPU time vs. N curve does not pass through the origin because a certain amount of "overhead" is involved in initiating each solution. (2) The cost of computation of elemental stiffness matrices and load vectors increases linearly with the number of elements and with the square p. (3) The cost of solution of the system of simultaneous equations increases approximately with the square of the number of degrees of freedom. For very large problems this cost overwhelms all other costs. (4) The cost ratio between h- and p-version solutions depends on the strongest singularity when quasiuniform meshes and p-distributions are used. In general, the cost ratio is more favorable for the p-version when the singularity is stronger. In this example the singularity was relatively mild. Had the problem contained an obtuse corner as in the case of the rhombic plate discussed in Section 5.5, the cost ratio would have been extremely high.

8. CONCLUSIONS AND RECOMMENDATIONS

Adaptive finite element technology has the potential for reducing the cost of structural stress analysis by about an order of magnitude. This will strongly impact structural synthesis where one of the current limitations is the cost of analysis.

Adaptivity can be based on either the h- or p-versions of the finite element method, or a combination of both. From the practical point of view, the goal being to obtain reliable approximations accurate to within one and three percent relative error in energy, adaptivity based on the p-version is the most promising approach. In particular, the p-version offers greater reliability through its robust behavior and greater efficiency (due to the fact that progressive mesh refinement is not required) than the h-version. Within the accuracy range required in engineering analysis, combination of the h- and p-versions does not offer practical advantages.

Full utilization of the adaptive process in structural synthesis requires that local a-posteriori error estimators be developed in those norms in which the constraints are specified. At present only preliminary results are available for the p version and only in energy norm. The state of the art is more advanced in the case of the h-version. We recommend that research should be vigorously pursued in this area and estimate that local a posteriori error estimators can be developed in 2 to 5 years. The estimated payoff would be approximately 50 percent reduction in cost over presently available adaptive approaches.

Certain adaptive strategies are already available. We refer to the fact that reasonably designed fixed meshes, coupled with uniform p-distribution, with p sufficiently high to guarantee the desired level of precision, can already result in very substantial reductions in the

cost of analysis. The amount of reduction is problem-dependent, nevertheless for practical problems it can be as high as several orders of magnitude, when compared with conventional finite element technology.

The choice of p-level must be based on considerations of the desired level of precision. In the absence of suitable a posteriori estimators, indirect error estimation should be used. Indirect error estimation is based on the fact that the parameters controlling the accuracy of solution in the p-version can be determined before starting the synthesis process.

An important advantage of the p-version, not previously realized, is its robustness. Examples were presented which indicated that the p-version is insensitive to Poisson's ratio in the case of elasticity and to plate thickness in the case of the Reissner-Mindlin theory of plates. This ensures proper convergence even when the plate or shell thickness ranges between wide limits in the structural synthesis process.

It has been shown that the Reissner-Mindlin theory of plates is capable of approximating the three dimensional elasticity solution to within one or two percent relative error in energy for all length-to thickness ratios and loading conditions likely to occur in practical applications of structural synthesis. The same statement cannot be made of the Kirchhoff theory of plates. For this reason, use of the Reissner-Mindlin theory is recommended for practical analyses. We remark, however, that this conclusion and recommendation are limited to plates manufactured from isotropic materials. Composite plates may present convergence problems unless the shear factor is carefully chosen.

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